NOTES FOR 20 OCT (FRIDAY)

1. Recap

(1) Defined the tensor product of vector bundles (in particular, defined the bundles of covariant, contravariant, and mixed tensors).
(2) Looked at how tensor fields look like locally (and how they change when you change coordinates).
(3) Recalled some linear algebra. Most importantly, we saw that $Map(V,W) \simeq W \otimes V^*$ and defined the contraction operation that took (k,l) tensors to $(k-1,l-1)$ tensors.

2. Tensors and tensor fields

Most of the above linear algebra can be recast in the index notation we have been using so far. When it comes to contractions, you can see the advantage of Einstein's notation.

(1) $V \otimes W \simeq W \otimes V$: Suppose $A = A^{i\mu} e_i \otimes f_\mu$. Then $T(A) = A^{i\mu} f_\mu \otimes e_i$ is the isomorphism, i.e., $(T(A))^{\mu i} = A^{i\mu}$.
(2) $(V \otimes W)^* \simeq V^* \otimes W^*$: $A_{i\mu}(e \otimes f)^{i\mu *} \to A_{i\mu}(e^i)^* \otimes (f^\mu)^*$, i.e., $A_{i\mu} \to A_{i\mu}$
(3) If $T = T_{i_1 i_2 \ldots i_k}(e^{i_1})^* \otimes (e^{i_2})^* \ldots$ and $S = S_{j_1 j_2 \ldots j_l}(e^{j_1})^* \otimes (e^{j_2})^* \ldots$ then $T \otimes S = T_{i_1 \ldots i_k} S_{j_1 \ldots j_l}(e^{i_1})^* \otimes \ldots$, i.e., $(T_I, S_J) \to T_I S_J$ (we sometimes use capital letters to denote multi-indices).
(4) $Map(V,W) \simeq W \otimes V^*$: If $T \in Map(V,W)$, then given a basis e_i for V and f_μ for W, a basis for $Map(V,W)$ is $h^i_\mu(v) = v^i f_\mu$. In this basis, $T = T^\mu_i h^i_\mu$. The isomorphism is given by $G(T^\mu_i h^i_\mu) = T^\mu_i f_\mu \otimes (e^i)^*$, i.e., $(G(T))^\mu_i = T^\mu_i$.
(5) $Cont : V \otimes V^* \to \mathbb{R}$. If $A = A^i_j e_i \otimes (e^j)^*$, then $Cont(A) = A^i_i$. More generally, if you have $A^{abc\ldots}{}_{\alpha\beta\gamma\ldots}$ then $Cont^3_1(A)$ will be $A^{abc\ldots}{}_{c\beta\gamma\ldots}$. This is one of the points of using the Einstein notation. You can keep track of contractions easily. (This is not some abstract requirement. If you study Riemannian geometry, you have to do contractions all the time. For example, the Ricci tensor is a contraction of the Riemann curvature tensor.)

Now we can translate these kinds of statements to vector bundles V, W (in particular, tensor bundles) on a manifold.

(1) First of all, note that any function $\mathcal{A} : Vect \to Func$ that is linear over smooth functions induces a unique smooth 1-form ω. Indeed, suppose $X_p \in T_pM$. Choose a coordinate system (x, U) and extend X_p to a smooth vector field on U as $X = (X_p)^i \frac{\partial}{\partial x^i}$. Suppose $\rho : U \to \mathbb{R}$ is a smooth bump function equal to 1 on p, then $Y = \rho X$ extends X_p to a smooth vector field on M. Define $\omega_p : T_pM \to \mathbb{R}$ as $\omega_p(X_p) = \mathcal{A}(X)$.
   (a) ω_p is independent of choices : It is enough to prove that $\mathcal{A}(X)(p) = 0$ if $X(p) = 0$.
   Firstly, note that if $Y = Z$ on any neighbourhood U of p, then $\mathcal{A}(Y)(p) = \mathcal{A}(Z)(p)$ (this property is called *locality* by physicists). Indeed, suppose $f : M \to \mathbb{R}$ is a bump function equal to 1 on p and with support in U, then $fY = fZ$ on all of M. Thus $\mathcal{A}(fY)(p) = \mathcal{A}(fZ)(p)$ which by linearity implies what we want.
   Secondly, let $Y = \rho X$. Now Y and X coincide on U and therefore $X(p) = \mathcal{Y}(p)$. Thus $\mathcal{A}(Y)(p) = \mathcal{A}(\rho X^i \frac{\partial}{\partial x^i})(p) = X^i(p)\mathcal{A}(\rho \frac{\partial}{\partial x^i})(p) = 0$ because $\rho \frac{\partial}{\partial x^i}$ is a global vector field and hence linearity of $\mathcal{A}$ can be used.

(b) It is a 1-form, i.e., $\in T_p^*M : \omega_p(aX_p + bZ_p) = \mathcal{A}(aX + bZ) = a\mathcal{A}(X)(p) + b\mathcal{A}(Z)(p) = a\omega_p(X_p) + b\omega_p(Z_p)$.

(c) It defines a smooth 1-form field as p varies : We just need to show that $\omega_p(\frac{\partial}{\partial x^i}(p))$ is smooth as a function of p for any coordinate chart (x, U). Indeed, take any $\rho : U \to \mathbb{R}$ which is smooth in a small neighbourhood of p and whose support is in U. Then $\omega_p(\frac{\partial}{\partial x^i}(p)) = \mathcal{A}(\rho\frac{\partial}{\partial x^i})(p)$ is a smooth function of p.

(2) If we are given a covariant tensor field of rank k, i.e., a smooth section A of $T^*M \otimes T^*M \ldots T^*M$, then we can get a multilinear map $\bar{A}$ that takes smooth vector fields $X_1, \ldots, X_k$ to smooth functions. Indeed, $\bar{A}(X_1, \ldots, X_k)(p) = A(p)(X_1(p), X_2(p), \ldots)$ where we are interpreting $A(p)$ as a multilinear map on T_pM. Actually, $\bar{A}$ is multilinear over smooth functions as well (because it is defined pointwise).

(3) The converse to the above also holds :

Theorem 2.1. *If $\mathcal{A} : Vect \times Vect \times \ldots \to Func$ is linear over smooth functions, then there is a unique smooth tensor field A with $\bar{A} = A$.*

Proof. It is the same proof as above but with multilinearity instead of linearity. □

(4) $V \otimes W \simeq W \otimes V$: On every fibre, linearly extend the map $v_p \otimes w_p \to w_p \otimes v_p$. This is a fibrewise isomorphism. We just need to prove that it is smooth. Indeed, upon choosing trivialisations e_i, f_μ for V, W, the map is $(p, A^{i\mu}) \to (p, A^{i\mu})$ which is smooth as a map from $U \times \mathbb{R}^v \otimes \mathbb{R}^w \to U \times \mathbb{R}^w \otimes \mathbb{R}^v$.

(5) $(V \otimes W)^* \simeq V^* \otimes W^*$: Similar to the above isomorphism. Another way of looking at this is through transition functions. I leave it as an exercise to show that for two invertible matrices A, B, $((A \otimes B)^{-1})^T = (A^{-1})^T \otimes (B^{-1})^T$.

(6) Any smooth bundle map $T : V \to W$ induces a unique smooth section of $W \otimes V^*$ and vice-versa : This will be given as HW.

(7) $Cont_i^j : T_l^k M \to T_{l-1}^{k-1} M$ is a bundle morphism : Indeed, fibrewise, applying the contraction map defined earlier. It is smooth because at the level of a local trivialisation, it is simply a linear map (summation of certain components of a multi-indexed object).

Lastly, given a smooth map $f : M \to N$, we have the pushforward $f_* : T_pM \to T_{f(p)}N$ and the pullback $f^* : T^*_{f(p)}N \to T^*_pM$. These induce linear maps $f_* : T_pM \otimes T_pM \otimes \ldots \to T_{f(p)N} \otimes \ldots$ as $f_*(v_1 \otimes v_2 \ldots) = f_*v_1 \otimes f_*v_2 \ldots$ and likewise for f^*. Just as in the usual case, covariant tensor fields (things like 1-forms) can be pulled back but not contravariant ones (things like vector fields).

3. Differential forms

One important goal is to generalise the fundamental theorem of calculus to higher dimensions, i.e., to write something like "Integral over a region R of some sort of derivative of some object A = Integral over the boundary of R of some object that depends on A (but not on its derivative)". For example, Green's theorem $\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dxdy = \int_{\partial R}(Pdx + Qdy)$. Likewise, we have the divergence theorem $\int\int\int_R \nabla.\vec{F}dV = \int\int_{\partial R} \vec{F}.d\vec{A}$ and the curl theorem $\int\int_R \nabla \times \vec{F}.d\vec{A} = \int_{\partial R} \vec{F}.d\vec{l}$. Why do we care about these generalisations ? one reason is that they help us study solutions of PDE (and also to come up with solutions in the first place). For example, if you take $\Delta u = f$ on $R \subset \mathbb{R}^3$ with $u = 0$

on ∂R, then multiplying by u and integrating we see that

$$
\begin{aligned}
&\int\int\int_R u\Delta u = \int\int\int uf \\
&\Rightarrow \int\int\int_R (\nabla.(u\nabla u) - |\nabla u|^2) = \int\int\int uf \\
&\Rightarrow \int\int_{\partial R} u\nabla u.d\vec{A} - \int\int\int |\nabla u|^2 = \int\int\int uf \\
&\Rightarrow -\int\int\int |\nabla u|^2 = \int\int\int uf \qquad (3.1)
\end{aligned}
$$

So we know that the derivative of such a solution is in some sense controlled by the solution itself.

So, yes, it is useful to generalise the fundamental theorem of calculus (in the style of the Green theorem) to higher dimensions. Obviously, the "region" in space should be replaced with a manifold-with-boundary. The real questions are "What sort of objects should we integrate ?", "What sort of derivatives should we take ?"