NOTES FOR 21 AUG (MONDAY)

1. Recap

(1) Proved a form of the Whitney embedding theorem for compact manifolds.
(2) Proved the existence of partitions of unity (both kinds).

2. Sard's theorem

Recall that we noticed in our examples that the set of critical *values* of a smooth map $f : M \to N$ seemed to be "small" (i.e. if you throw a dart at N, it is unlikely that you will hit at a critical value. Moreover, regular values are dense). This is a general phenomenon called Sard's theorem. To make the notion of "small" precise, we need to know what "measure zero" means on a manifold. One way is to define the concept of measure on N using a Riemannian metric (more precisely, using the volume form of the same). But all we need to define is "measure zero". This concept does not need the full definition of a measure. Naively, "a set $A \subset N$ is of measure 0 if we can cover it with countably many measure 0 coordinate sets". But to make sense of this definition, we need to know how measure changes under diffeomorphisms (i.e. is this definition independent of the coordinates chosen ?). To this end, we need a couple of lemmata :

Lemma 2.1. *Let $A \subset \mathbb{R}^n$ be a rectangle and $f : A \to \mathbb{R}^n$ be a C^1 function such that $|D_i f^j| \leq K \; \forall \; i, j$. Then $|f(x) - f(y)| \leq n^2 K |x - y|$.*

Proof.

$$|f(x) - f(y)| \leq \sum_j |\int_0^1 \frac{df^j \circ (tx + (1-t)y)}{dt} dt| = \sum_j \int_0^1 |D_i f^j||(x^i - y^i)| dt$$

$$\leq \sum_{j,i} K|x - y| = n^2 K |x - y|$$

□

Lemma 2.2. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $A \subset \mathbb{R}^n$ has measure 0, then so does $f(A)$ have measure 0.*

Proof. Since $\mathbb{R}^n$ is a countable union of compact sets K_i, take $A \cap K_i$. This has measure 0. We shall prove that $f(A \cap K_i)$ has measure zero (thus proving the desired result).

By compactness, there exists a K such that lemma 2.1 applies. This means that if cover A with countably many rectangles whose total measure is ϵ, then $f(A)$ can be covered by rectangles whose total measure is at most $\epsilon \times (n^2 K)^n$. This proves the desired result. □

Now we *define* a subset $A \subset M$ of a smooth manifold to have measure 0 if $A \subset \cup_{i=1}^{\infty} U_i$ where U_i are countably many coordinate charts such that each set $x_i(A \cap U)$ has measure 0. This notion is easily seen by the above lemmata to be independent of the coordinates chosen.

Moreover, if $A \cap U$ has measure 0 for every coordinate chart, and M is second countable, then A has measure 0. (If M is the union of uncountable many $\mathbb{R}$s, then this is clearly false.)

As a corollary we have

Corollary 2.3. *If $f : M \to N$ is a C^1 map and $A \subset M$ has measure 0, then so does $f(A)$.*

Proof. $A \subset \cup_{i=1}^{\infty} U_i$ such that $A \cap U_i$ is of measure 0. Now for each chart $(y, V) \subset N$, $f(A) \cap V$ has measure 0 by lemma 2.2. Since $f(\cup_{i=1}^{\infty} U_i)$ is contained in at most countably many components of N, $f(A)$ has measure 0. □

Finally we have

Theorem 2.4 (Sard). *If $f : M \to N$ is a C^k map, and M has countably many components, then the critical values are a set of measure 0 if $k \geq 1 + \max(m - n, 0)$.*

The proof of this theorem is quite non-trivial. However, it is easily seen that (using coordinate charts) it can be reduced to the case when M and N are $\mathbb{R}^m, \mathbb{R}^n$. The proof is in Milnor's book. A special case is easy to prove :

Lemma 2.5. *If $f : M^m \to N^n$ is C^1 and $n > m$ then $f(M)$ has measure 0 if M has only countably many components.*

Proof. Firstly, it is easy to see (why?) that we just need to prove this for $\mathbb{R}^m$ to $\mathbb{R}^n$.
Secondly, cover $\mathbb{R}^m$ with countably many closed cuboids R_i of size $\epsilon < 1$ each. Let K_i be the constant (which exists by compactness) necessary to apply lemma 2.1. Now $|f(x) - f(y)| \leq mnK_i|x - y|$ by the proof of that lemma. The measure of $f(R_i)$ is at most $(mnK_i)^n \epsilon^n = C\epsilon^n$. Let $\mathbb{R}^m = \cup_i U_i$ where $U_i = [-i, i]^m$. In order to cover U_i, one needs at most $\frac{(2i)^m}{\epsilon^m}$ number of cuboids. Thus the measure of $f(U_i)$ is at most $\tilde{C}\epsilon^{n-m}$. By choosing ϵ to be arbitrarily small, it is seen that $f(U_i)$ has measure 0 $forall\ i$. Thus $f(M)$ has measure 0. □