NOTES FOR 22 NOV (WEDNESDAY)

1. Recap

(1) Defined inner product, volume form, and the induced inner product on V^*.
(2) Defined a metric on a vector bundle, proved that metrics exist, and as a corollary, $V \simeq V^*$.

2. Riemannian geometry

In the special case when $V = TM$, the metric is called a Riemannian metric on M. If (x, U) is a coordinate chart, then $g(x) = g_{ij}(x)dx^i \otimes dx^j$. By symmetry, $g_{ij} = g_{ji}$. Moreover, g is a positive definite matrix. If one changes coordinates to y^μ then $g_{\mu\nu} = g_{ij}\frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu}$. Given a metric g on TM, we get one on T^*M given by $g^* = g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Now $g_{ik}g^{kj} = \delta^j_i$.

If M is oriented, supposing (x, U) is an oriented coordinate chart, then $vol = \sqrt{\det(g_{ij})}dx^1 \wedge dx^2 \ldots dx^m$ is a well-defined top form. Indeed, if we changes coordinates, it transforms correctly as seen in the linear algebra above. This is called the "volume" form of the metric.

Here are examples :

(1) Euclidean space $\mathbb{R}^n, g_{Euc} = \sum dx^i \otimes dx^i$. This is the usual metric. The length of a tangent vector v is $\sum(v^i)^2$.
(2) If we take the same Euclidean space $\mathbb{R}^2$ and use polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, then $dx = dr\cos(\theta) - r\sin(\theta)d\theta$, $dy = dr\sin(\theta) + r\cos(\theta)d\theta$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta$. This automatically raises the question answered by Riemann "Suppose we write some metric in $\mathbb{R}^n$, how do we know that it is not just the Euclidean metric written in fancy coordinates ?"
(3) The circle $S^1 : g = d\theta \otimes d\theta$.
(4) If M, g_M, N, g_N are two Riemannian manifolds, then $M \times N, g_M \times g_N$ given by $g_M \times g_N(v_M \oplus v_N, w_M \oplus w_N) = g_M(v_M, w_M) + g_N(v_N, w_N)$. This gives a metric on the n-torus using the circle metric.
(5) The Hyperbolic metric $\mathbb{H}^m, g_{Hyp} : g_{Hyp} = \frac{\sum dx^i \otimes dx^i}{(x^m)^2}$.

The above examples still don't tell us to how construct a metric on the sphere for instance. So we introduce the following definition :

Definition 2.1. If g is a metric on M and $S \subset M$ is an embedded submanifold, then g induces a metric $g|_S$ on S given by $g_p|_S(v_S, w_S) = g_p(i_* v_S, i_* w_S)$.

In coordinates, suppose $y^1 = x^1, \ldots, y^s = x^s$ are coordinates on S and $x^{s+1}, \ldots, x^m$ are functions of these on S, then $g|_S = g_{ij}\frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu}dy^\mu \otimes dy^\nu$. Now we can write down lots of examples.

(1) $S^n \subset \mathbb{R}^{n+1}$. Suppose we choose the coordinate chart where $x^{n+1} > 0$, then $x^{n+1} = \sqrt{1 - \sum(x^i)^2}$. Thus, $g_{Sphere} = \sum dx^i \otimes dx^i + \frac{\sum_j x^j dx^j}{\sqrt{1-\sum(x^i)^2}} \otimes \frac{\sum_k x^k dx^k}{\sqrt{1-\sum(x^i)^2}}$. In the special case of S^2, this boils down to $g = \frac{1}{1-(x^1)^2-(x^2)^2}[(1-(x^2)^2)dx^1 \otimes dx^1 + (1-(x^1)^2)dx^2 \otimes dx^2 + x^1x^2(dx^1 \otimes dx^2 + dx^2 \otimes dx^1)]$. This is pretty complicated. A simpler way to write the metric is using another coordinate chart, i.e., first write the metric in $\mathbb{R}^3$ in spherical coordinates $z = r\cos(\theta)$,

$x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2(\theta) d\phi \otimes d\phi$. Now when we restrict to the unit sphere, the tangent vectors do not include $\frac{\partial}{\partial r}$. Thus, $g_{Sphere} = d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi$

(2) Suppose $z = f(x, y)$ is the graph of a function, then $g_{Induced} = dx \otimes dx + dy \otimes dy + (\frac{\partial f}{\partial x})^2 dx \otimes dx + (\frac{\partial f}{\partial y})^2 dy \otimes dy + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(dx \otimes dy + dy \otimes dx)$.

(3) $GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ has the Euclidean metric.

Now we write down the volume forms of most of the above examples :

(1) $vol_{Euc} = dx^1 \wedge dx^2 \wedge \ldots dx^n$.

(2) In polar coordinates in $\mathbb{R}^2$, $vol_{Euc} = \sqrt{\det(g)} dr \wedge d\theta = r dr \wedge d\theta$.

(3) For the circle, $vol = d\theta$.

(4) If we take the product metric on $M \times N$, then $vol = \pm vol_M \wedge vol_N$ (depending on the orientation chosen).

(5) $vol_{Hyp} = \frac{1}{(x^m)^m} dx^1 \wedge dx^2 \ldots dx^m$.

(6) $vol_{S^n} = \frac{1}{\sqrt{1 - \sum (x^i)^2}} dx^1 \wedge dx^2 \ldots dx^n$.

(7) $vol_{graph} = \sqrt{\det(g)} dx \wedge dy = \sqrt{(1 + (\frac{\partial f}{\partial x})^2)(1 + (\frac{\partial f}{\partial y})^2) - (\frac{\partial f}{\partial x}\frac{\partial f}{\partial y})^2} dx \wedge dy = \sqrt{1 + \frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2} dx \wedge dy$