# NOTES FOR 23 AUG (WEDNESDAY)

## 1. Recap

(1) Covered Sard's theorem. Defined measure zero.
(2) By the way, you can convince yourself that Sard's theorem is not unreasonable by looking at simple maps like $(x, y) \to (ax + by, cx + dy)$ and $(x, y) \to (x^2, y^2)$.

## 2. Vector fields, Tangent bundle, Cotangent bundle, etc

Remember that we had a natural question when we defined the "partial derivatives" of a map $f : M \to \mathbb{R}$. This notion required the choice of a coordinate chart and hence there is no meaning to $df : M \to \mathbb{R}^m$. However, our question was "Is there some other manifold $N$ such that $df : M \to N$ makes sense ?" Among other things, we shall answer that question here. (Spoiler : $N$ will be denoted (later) as $T^*M$ and be called the cotangent bundle of $M$.) Why bother finding such a manifold $N$ ? So that we can apply all our machinery developed for manifolds (like the constant rank theorem and Sard's theorem for instance) to $N$.

Indeed, if we choose a coordinate chart $(x, U)$, then $df : U \to \mathbb{R}^m$ is $df(p) = (\frac{\partial f}{\partial x^1}(p), \ldots)$. But if we choose a different coordinate chart $(y, V)$, then on $U \cap V$, the two $df$'s are not the same. They change as $w_i = \frac{\partial f}{\partial y^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial y^i} = v_j \frac{\partial x^j}{\partial y^i}$ in accordance with the chain rule. So one can attempt to construct a manifold by taking the following weird quotient $N = \dfrac{\coprod_\alpha U_\alpha \times \mathbb{R}^m}{(\vec{x}_\alpha, \vec{v}_\alpha) \sim (\vec{x}_\beta, \vec{v}_\beta) \ \Leftrightarrow \ \Phi_\alpha^{-1}(\vec{x}_\alpha) = \Phi_\beta^{-1}(\vec{x}_\beta) \ and \ (v_\alpha)_i = \frac{\partial x_\beta^j}{\partial x_\alpha^i}(v_\beta)_j}$ where $U_\alpha$ forms an atlas for $M$ (where each $U_\alpha$ is homeomorphic to all of $\mathbb{R}^m$). (Why is this a genuine equivalence relation?)

Here is a sketch of the proof that $N$ is indeed a smooth manifold of dimension $2n$ and $df : M \to N$ is a well-defined smooth map : Let $\pi : \coprod_\alpha U_\alpha \times \mathbb{R}^m \to N$ be the quotient map. $N$ is Hausdorff, paracompact, and $\pi$ is an open map (Why?). Consider the open cover of $N$ formed by the open sets $\tilde{U}_\alpha = \pi(U_\alpha \times \mathbb{R}^m)$. Now here are coordinate charts $\tilde{\Phi}_\alpha : \tilde{U}_\alpha \to \mathbb{R}^{2m}$ given by $\tilde{\Phi}_\alpha(\pi(x_\alpha, v_\alpha)) = (x_\alpha, v_\alpha)$. (Why are these homeomorphisms ?) The transition functions are $\tilde{\Phi}_\alpha \circ \tilde{\Phi}_\beta^{-1}(x_\beta, v_\beta) = (x_\alpha, \frac{\partial x_\beta^j}{\partial x_\alpha^i}(v_\beta)_j)$ which are diffeomorphisms (Why? Hint : Note that the second factor is simply an invertible linear map (varying smoothly) that takes $v_\beta$ to $v_\alpha$). So $N$ is hausdorff, paracompact, and locally euclidean (with the transition functions being smooth). Thus it is a smooth manifold with the smooth structure being induced by the unique maximal atlas containing the one that we provided. Moreover, define $df : M \to N$ as follows : Suppose $p \in U_\alpha$, then $df(p) = \pi(x_\alpha(p), \frac{\partial f}{\partial x_\alpha^1}(p), \ldots)$. It is easy to check that this is well-defined and smooth.

Note that $N$ has the following properties :

(1) $N$ is a smooth manifold of dimension $2m$ (we will denote it as $T^*M$ from now onwards).
(2) There is a smooth map (which by an unfortunate choice of notation we will denote as $\pi$, but please do not confuse this with the previously denoted quotient map used to construct $N$) $\pi : N = T^*M \to M$ such that $\pi^{-1}(p)$ is a real vector space of dimension $m$.

(3) $\pi$ satisfies the "local triviality" condition : Around every point $p \in M$, there is an open neighbourhood $V_p$ and a diffeomorphism $\phi : \pi^{-1}(V_p) \to V_p \times \mathbb{R}^m$ such that $\pi_1 \circ \phi = \pi$ and $\phi$ is a linear map on the second factor.
(4) The map $df : M \to T^*M$ satisfies $\pi \circ df(p) = p$.

We will see objects like $N$ having these properties appear over and over again in the study of manifolds (and for that matter, algebraic geometry, and even physics). So it is best to "abstract out" these properties into a definition once and for all :

**Definition 2.1.** A rank-$r$ smooth real vector bundle over a smooth manifold $M$ is a tuple $(E, \pi, M, \oplus, \odot)$ where

(1) $E$ is a smooth manifold of dimension $m + r$. It is called "the total space of the bundle". $M$ is called "the base space of the bundle".
(2) $\pi : E \to M$ is a smooth onto $M$.
(3) $\oplus$ and $\odot$ are maps $\oplus : \cup_{p\in M}\pi^{-1}(p) \times \pi^{-1}(p) \to E$ and $\odot : \mathbb{R} \times E \to\subset \pi^{-1}(p)E$ such that $\oplus(\pi^{-1}(p) \times \pi^{-1}(p)) \subset \pi^{-1}(p)$ and $\odot(\mathbb{R} \times \pi^{-1}(p)) \subset \pi^{-1}(p)$ make each "fibre" $\pi^{-1}(p)$ into an r-dimensional vector space over $\mathbb{R}$.

such that the "local triviality" condition is satisfied : For each $p \in M$ there is an open neighbourhood $V_p$ and a diffeomorphism $\phi : \pi^{-1}(U) \to V_p \times \mathbb{R}^r$ such that $\pi_1 \circ \phi = \pi$ and $\pi$ is a vector space isomorphism from each $\pi^{-1}(q)$ onto $q \times \mathbb{R}^r$ for all $q \in U$.

By abuse of notation, whenever we say "vector bundle", we will denote it as $E$ (and forget about the tuple).

The map $df : M \to T^*M$ has a special property which will make into a definition :

**Definition 2.2.** Suppose $E$ is a vector bundle over $M$. A smooth map $s : M \to E$ is called a section if $\pi \circ s(p) = p$.

It is clear that every vector bundle has a "stupid section", namely the zero section $s_0$ which is defined by requiring $s_0(p)$ to be 0 in the vector space $\pi^{-1}(p)$ lying above $p$, i.e., for every local trivialisation $\phi$, $\phi \circ s_0(p) = (p, \vec{0})$. I leave it as an exercise to check that $M$ is embedded in $V$ through the zero section $s_0$.

Note that if $U$ is a coordinate open set, then $dx^i : U \subset M \to T^*M$ are local smooth sections, i.e., smooth maps such that $\phi \circ dx^i(p) = p$. Note that $dx^i(p) = [p, \frac{\partial x^i}{\partial x^1}, \ldots] = [p, 0, \ldots, 1, 0, \ldots]$. Moreover, notice that smooth sections form a vector space. Indeed, given two sections $s_1, s_2$, we define their addition $(s_1 + s_2)(p) = [p, \pi_2 \circ s_1(p) + \pi_2 \circ s_2(p)]$ and likewise, scalar multiplication. The $dx^i(p)$ form a basis for the fibre $\pi^{-1}(p)$ for all $p \in U$ (indeed, in the $x^i$ coordinates, they correspond to the standard basis of $\mathbb{R}^m$). Also, it is easy to see that every smooth section $\omega : M \to T^*M$ is locally, on a coordinate open set $U$, of the form $\omega(p) = [p, \omega_i(p)dx^i(p)]$ where $\omega_i : U \to \mathbb{R}$ are smooth functions. Under change of coordinates, they transform as $\omega_{i,\alpha} = \frac{\partial x^j_\beta}{\partial x^i_\alpha}\omega_{j,\beta}$. Sections of $T^*M$ are called "1-forms".