NOTES FOR 23 OCT (MONDAY)

1. Recap

(1) Wrote down the linear algebra properties of tensor products in the index notation.
(2) Generalised some properties to vector bundles. In particular, proved that given a multilinear function that takes vector fields to functions, we get a unique 1-form field.
(3) Motivated the question of generalising the fundamental theorem of calculus to higher dimensions, i.e., versions of the Green, divergence, and curl theorems.

2. Differential forms

Looking at the above examples, it seems that the notion of a cross product plays an important role. Indeed, $d\vec{A}$ and $\nabla \times \vec{F}$ involve cross products. Naively, if we want to extend cross products of vectors to $\mathbb{R}^4$, then the components of $\vec{a} \times \vec{b}$ will be $a_i b_j - a_j b_i$ where $1 \leq i, j \leq 4$. In other words, there are more than 4 independent components! So whatever $\vec{a} \times \vec{b}$ is, it is definitely not a vector in $\mathbb{R}^4$! Perhaps the correct way to talk about cross products is determinants. Maybe it only makes sense to talk about

$$\vec{a}\text{“}\times\text{”}\vec{b}\text{“}\times\text{”}\vec{c} = \begin{vmatrix} i & j & k & l \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \tag{2.1}$$

So “$\nabla \times \vec{F}$” should be replaced with $(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k + \frac{\partial}{\partial w} l)$“$\times$”$F$ where F is no longer a vector but an object involving 2-indices.

All of these considerations motivate us to study “differential forms”, i.e., tensors that are anti-symmetric/alternating. An alternating tensor $T \in \mathcal{T}^k(V^*)$ corresponds to a multilinear map such that $T(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$. It is easy to see (using $v_i + v_j$ in the places of v_i, v_j) that this is equivalent to being anti-symmetric, i.e. $T(v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots)$. (This is only for fields of characteristic $\neq 2$.) We denote alternating tensors as $\Omega^k \subset \mathcal{T}^k$. Note that $\Omega^0 = \mathbb{R}$ and $\Omega^1 = \mathcal{T}^1$. Note that if $S : V \to W$ is a linear map, then $S^* : W^* \to V^*$ induces a map $S^* : \mathcal{T}^k(W^*) \to \mathcal{T}^k(V^*)$. It is easy to see that this preserves Ω^k and hence induces a map between the alternating tensors. The simplest example of an alternating tensor is the determinant. Here are two useful little points :

(1) $\dim(\Omega^k(V^*)) = 0$ if $k > dim(V)$: Indeed, evaluate this on a basis of $V \otimes V \otimes \ldots$. For each of the basis vectors, at least one of the e_i repeats.
(2) $\dim(\Omega^{dim(V)}(V^*)) = 1$: $\omega(v_1, v_2, \ldots, v_n) = v_1^{i_1} v_2^{i_2} \ldots v_n^{i_n} \omega(e_{i_1}, \ldots, e_{i_n})$. If the indices repeat, then $\omega(v_1, \ldots, v_n) = 0$. If not, note that any permutation of $e_1, \ldots, e_n$ can be accomplished by a sequence of transpositions. Hence $\omega(e_{i_1}, \ldots)$ is upto a sign, the same as $\omega(e_1, \ldots, e_n)$. Thus, every ω is a multiple of the determinant map.

Actually, it is not that hard to construct examples of alternating tensors. As an analogy (which is more than an analogy as we will see), suppose I give you an $n \times n$ matrix A_{ij}. Is there a natural way to construct an anti-symmetric matrix from it ? Sure, take $\frac{A_{ij} - A_{ji}}{2}$, i.e., “antisymmetrize” the

indices. More generally, one does the following :
Let σ be a permutation of $1, \ldots, k$, i.e., it is a function $i \to \sigma(i)$. If $(v_1, \ldots, v_k)$ is a tuple of any objects, we define $\sigma.(v_1, \ldots, v_k) = (w_1, \ldots, w_k) = (v_{\sigma(1)}, \ldots, v_{\sigma(k)})$. This is sort of like a group action in the sense that $\sigma.(\rho.(v_1, \ldots, v_k)) = (\rho\sigma).(v_1, \ldots, v_k)$. Now we define the antisymmetrization or the alternation of a tensor $T \in \mathcal{T}^k$ as

$$AltT = \frac{1}{k!} \sum_{\sigma \in S_k} sign(\sigma) T.\sigma \tag{2.2}$$

i.e., $AltT(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sign(\sigma) T(v_{\sigma(1)}, v_{\sigma(2)}, \ldots)$.
Here is a proposition :

Proposition 2.1. (1) *If $T \in \mathcal{T}^k$, then $Alt(T) \in \Omega^k$*
(2) *If $\omega \in \Omega^k$ then $Alt(\omega) = \omega$ (We need the $\frac{1}{k!}$ for this property)*
(3) *If $T \in \mathcal{T}^k$ then $Alt(Alt(T)) = Alt(T)$.*

Proof. (1) We need to prove that if we interchange (transpose) two vectors, then $Alt(T)$ acting on the new set picks up a negative sign. Indeed, $AltT \circ tran = \frac{1}{k!} \sum sgn(\sigma).T \circ \sigma \circ tran = \frac{1}{k!} sgn(tran) \sum sgn(\tilde{\sigma}).T \circ \tilde{\sigma} = -AltT$.
(2) $Alt\omega = \frac{1}{k!} \sum sgn(\sigma)\omega \circ \sigma = \frac{1}{k!} \sum sgn^2(\sigma)\omega = \omega$.
(3) Follows the previous one.

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Now we define the "cross" product, i.e., the wedge product $\wedge$ of two alternating tensors $\omega \wedge \eta$: $\omega \wedge \eta = \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta)$. It satisfies the following properties

1. $\wedge$ is bilinear
2. If $T : V \to W$, then $T^*(\omega \wedge \eta) = T^*\omega \wedge T^*\eta$.
3. $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$. So if ω is an odd alternating tensor (i.e. of odd rank), then $\omega \wedge \omega = 0$. On the other hand, this is not necessarily true for even tensors.

Actually, the wedge product (just like the cross product) is also associative, but that requires a little bit of effort to prove.

Theorem 2.2. *The following hold.*

1. *If $S \in \mathcal{T}^k$, $T \in \mathcal{T}^l$, and $Alt(S) = 0$, then $Alt(S \otimes T) = Alt(T \otimes S) = 0$.*
2. *$Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta))$.*
3. *$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!l!m!} Alt(\omega \otimes \eta \otimes \eta)$.*

Proof. Indeed,

(1)

$$Alt(S \otimes T)(v_1, \ldots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma} sgn(\sigma) S \otimes T(v_{\sigma(1)}, \ldots, v_{\sigma(k+l)})$$
$$= \frac{1}{(k+l)!} \sum_{\sigma} sgn(\sigma) S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \otimes T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \tag{2.3}$$

Any permutation that fixes the last l vectors obviously goes to 0. Call the subgroup of such permutations G. Consider the left cosets $\sigma_0 G$. Then

$$\frac{1}{(k+l)!}\sum_{\sigma^{'}\in G} sgn(\sigma_0\sigma^{'})S\otimes T\sigma^{'}.(v_{(\sigma_0(1))},\ldots,v_{\sigma_0(k)},v_{\sigma_0(k+1)},\ldots,v_{\sigma_0(k+l)})=0 \tag{2.4}$$

Since summation over the group can be broken up into summation over left cosets, we are done. Likewise, $Alt(T\otimes S)=0$.

(2) Note that $Alt(Alt(\omega\otimes\eta)-\omega\otimes\eta)=0$. Therefore, $Alt((Alt(\omega\otimes\eta)-\omega\otimes\eta)\otimes\theta)=0$. Likewise for the other equality.

(3) Note that

$$\begin{aligned}(\omega\wedge\eta)\wedge\theta &= \frac{(k+l+m)!}{(k+l)!m!}Alt((\omega\wedge\eta)\otimes\theta)\\ &= \frac{(k+l+m)!}{(k+l)!m!}\frac{(k+l)!}{k!l!}Alt(Alt(\omega\otimes\eta)\otimes\theta)\end{aligned}$$

By the previous part, we are done.

$\square$

If $\phi_1,\phi_2,\phi_3,\phi_4$ is a basis of a four dimensional dual vector space V^*, then $\omega=\phi_1\wedge\phi_2+\phi_3\wedge\phi_4$ is an alternating 2-tensor. Now $\omega\wedge\omega=2\phi_1\wedge\phi_2\wedge\phi_3\wedge\phi_4\neq 0$. Indeed note that $\phi_1\wedge\phi_2\wedge\phi_3\wedge\phi_4(e_1,e_2,e_3,e_4)=1$ - We need the weird factor in the definition of the wedge product for precisely this equality. More generally, if $\phi_1,\ldots,\phi_k$ is a basis, then $\phi_1\wedge\ldots\phi_k(e_1,\ldots,e_k)=1$. This alternating tensor is nothing but the determinant map.