NOTES FOR 24 NOV (FRIDAY)

1. Recap

(1) Defined Riemannian metric and its volume form.
(2) Wrote down several examples (especially of induced metrics).

2. Riemannian geometry

Now we write down one more volume form :

(1) $vol_{graph} = \sqrt{\det(g)} dx \wedge dy = \sqrt{(1+(\frac{\partial f}{\partial x})^2)(1+(\frac{\partial f}{\partial y})^2) - (\frac{\partial f}{\partial x}\frac{\partial f}{\partial y})^2} dx \wedge dy = \sqrt{1+\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2} dx \wedge dy$

The next order of business is to try to make (M, g) into a metric space by defining the length of a curve and finding the shortest length curves. Suppose $\gamma : [0, 1] \to M$ is a smooth path. Then, define its length as $L(\gamma) = \int_0^1 \sqrt{g(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})} dt$. So in Euclidean space $\mathbb{R}^2$, $L(\gamma) = \int_0^1 \sqrt{(x')^2 + (y')^2} dt$. It can be proven by elementary means that the shortest path between any two points is a straight line. In general, the shortest path may not even exist. (Take $\mathbb{R}^2 - 0$ for instance.) Let the energy $E = \int |\gamma'|^2 dt$. (Note that by Cauchy-Schwartz, $E \leq L^2$. There is a more refined relationship.) For now suppose $\gamma(t)$ is the smallest energy path joining $p = \gamma(0)$ and $q = \gamma(1)$. Then, suppose $\phi(s, t) : I \times I \to M$ be a smooth variation of γ keeping the endpoints fixed, i.e., $\phi(s, 0) = p, \phi(s, 1) = q$, $\phi(0, t) = \gamma(t)$. Hence $\frac{dE(\phi)}{ds}|_{s=0} = 0$. So we calculate the first variation

Theorem 2.1. $\frac{dE(\phi)}{ds}_{s=0} = -\int_0^1 g_{lr} \frac{d\phi^l}{ds}|_{s=0}(0, t)[\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}]$ *where* $\Gamma^r_{ij} = g^{rk}\frac{1}{2}(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k})$

Proof.

$$\frac{dE(\phi)}{ds}_{s=0} = \int \frac{\partial g_{ij}}{\partial x^k}\frac{d\phi^k}{ds}|_{s=0}(\gamma^i)'(\gamma^j)' + \int g_{ij}(\frac{d(\phi^i)'}{ds}|_{s=0}(\gamma^j)' + \frac{d(\phi^j)'}{ds}|_{s=0}(\gamma^i)')$$

Integrating the second term by parts gives the result. ☐

Corollary 2.2. *If $\gamma : [a, b] \to M$ is a smooth path, then γ is a critical point for E if and only if for every coordinate system (x, U) we have*

$$\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0 \quad (2.1)$$

Proof. Suppose $\gamma(t) \in U$. Choose a $[t_i, t_{i+1}]$ such that the image of γ lies in U. Using variations preserving $\gamma(t_i), \gamma(t_{i+1})$ as in the theorem above, we see that

$$\int_0^1 g_{lr}\frac{d\phi^l}{ds}|_{s=0}(0, t)[\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}] = 0 \quad (2.2)$$

We can easily find a variation ϕ such that $\frac{\partial\phi}{\partial s}|_{s=0}$ is 0 outside (t_{i-1}, t_i) but is a positive function times $\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}$ inside. Thus

$$\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0.$$

☐

By the existence uniqueness theorem for ODE, for each $p \in M, v_p \in T_pM$, there is locally a unique smooth energy-minimiser through p pointing along v_p. Energy-minimisers are called geodesics.
We can generalise the first variation formula to piecewise smooth paths and variations that preserve only the endpoints.

$$\frac{dE(\phi)}{ds}_{s=0} = \int \frac{\partial g_{ij}}{\partial x^k}\frac{d\phi^k}{ds}|_{s=0}(\gamma^i)'(\gamma^j)' + \int g_{ij}(\frac{d(\phi^i)'}{ds}|_{s=0}(\gamma^j)' + \frac{d(\phi^j)'}{ds}|_{s=0}(\gamma^i)')$$
$$- \sum_{i=0}^{N}\langle\frac{\partial\phi(0,t_i)}{\partial s}, \frac{d\gamma}{dt}(t_i^+) - \frac{d\gamma}{dt}(t_i^-)\rangle \tag{2.3}$$

As a corollary,

Corollary 2.3. *A piecewise smooth path $\gamma : [a,b] \to M$ is a critical point for the Energy if and only if γ is actually smooth and satisfies the geodesic equation on every coordinate system.*

Proof. Choosing the ϕ as before, we see that the geodesic equation is satisfied on $[t_{i-1}, t_i]$. By choosing the variation now such that $\frac{\partial\phi(0,t_i)}{\partial s} = \frac{d\gamma}{dt}(t_i^+) - \frac{d\gamma}{dt}(t_i^-)$ we see that the $\Delta_i\frac{d\gamma}{dt} = 0$. By local uniqueness and existence of smooth solutions, γ is smooth. □

It is easy to see that geodesics in Euclidean space are of the form $\gamma = \vec{A} + \vec{B}t$. Note that the length is reparametrisation invariant but not the energy. Indeed, it turns out that

Theorem 2.4. *If γ is a critical point of the energy, then γ is parametrised by the arc length.*

Proof. To show this is equivalent to showing that the speed is a constant. Indeed, a long calculation shows this. (In Spivak.) □

Lastly, a long calculation shows that the critical points for the length functional satisfy

$$\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} - \frac{d\gamma^k}{dt}\frac{d^2s/dt^2}{ds/dt} = 0 \tag{2.4}$$

where $s(t)$ is the arc-length. Now assuming that $\gamma' \neq 0$ throughout, we can arc-length parametrise it $\gamma \circ s^{-1}$. Such a curve will be a geodesic. It turns out that every critical point of the length functional can be parametrised in a way so that the speed is nowhere zero (there are no "kinks"). In other words, arc-length parametrised geodesics are precisely critical points of the length functional.

Lastly, since $E \leq L^2$, if a geodesic (parametrised by arc-length) minimises E, then since $E = L^2$ for such a geodesic, L is also minimised by it.

One can verify easily that vertical lines are circles passing through points on the plane $x^n = 0$ are geodesics for the Hyperbolic metric. In fact, by the existence and uniqueness theorem on geodesics, they are all the geodesics of this metric.