NOTES FOR 25 OCT (WEDNESDAY)

1. Recap

(1) Agreed that it is important to generalise the cross product to higher dimensions. Saw that antisymmetric matrices/more general objects appear.
(2) Defined alternating/skew-symmetric tensors. Constructed them using the usual tensors by the alternation map (with a weird factor to make sure that $Alt(\omega) = \omega$).
(3) Defined the wedge product (with a weird factor for associativity) and proved its properties. Most notably, associativity.
(4) Saw that $\omega \wedge \omega = 0$ if ω is of odd rank but not necessarily so if ω is of even rank. Also saw that $(e^1)^* \wedge \ldots (e^n)^* \neq 0$ and that it corresponds to the determinant tensor.

2. Differential forms

More generally, suppose $(e^1)^*, \ldots, (e^n)^*$ form a basis for V^*, then $\omega = (e^1)^* \wedge (e^2)^* \ldots (e^k)^* \neq 0$ for any k. Indeed, $\omega = \frac{(1+1\ldots+1)!}{1!\ldots 1!} Alt((e^1)^* \otimes \ldots)$. Therefore $\omega(e_1, \ldots, e_k) = 1$. Moreover, $\omega(e_{i_1}, \ldots, e_{i_k}) = 0$ if $I \neq \{1, 2, \ldots, k\}$. In fact, we have :

Theorem 2.1. *Suppose $(e^I)^* = (e^1)^*, \ldots, (e^n)^*$ form a basis for V^*, then the set of all $(e^{i_1})^* \wedge (e^{i_2})^* \ldots \wedge (e^{i_k})^*$ where $1 \leq i_1 < i_2 < \ldots < i_k$ form a basis for Ω^k. Therefore, $dim(\Omega^k) = \frac{n!}{(n-k)!k!}$.*

Proof. The argument above shows that $(e^I)^* \neq 0$. Suppose they are linearly dependent, i.e., there exist a_I (not all 0) such that $\omega = a_I (e^I)^* = 0$. Then $\omega(e_J) = a_I (e^I)^*(e_J) = a_I \delta^I_J = a_J = 0$. A contradiction.
Now we prove that they span Ω^k. Indeed, suppose $\omega \in \Omega^k$. Then define $a_I = \omega(e_I)$ (where I is increasing). I claim that $\eta = \omega - a_I(e^I)^* = 0$. Indeed, since it is enough to test on all basis vectors of the form $e_I = (e_{i_1}, e_{i_2}, \ldots)$ where I is increasing (because if we arrange in any other order, we simply pick up a sign), we see that $\eta(e_I) = a_I - a_I = 0$.
Alternatively, $\omega = a_I(e^I)^*$ where $(e^I)^* = e^{i_1} \otimes e^{i_2} \ldots$. Since $\omega = Alt(\omega)$ we see the result. ☐

and

Corollary 2.2. *If $\omega_1, \ldots, \omega_k \in \Omega^k$ then they are linearly independent if and only if $\omega_1 \wedge \omega_2 \ldots \omega_k \neq 0$.*

Proof. If ω_i are linearly independent, then we can extend them to a basis of V^* and use the previous result to conclude that $\omega_1 \wedge \omega_2 \ldots \omega_k \neq 0$.
If they are dependent, i.e., $\omega_1 = \sum_{i=2}^n a_i \omega_i$, then $(\sum a_i \omega_i) \wedge \omega_2 \ldots = a_2 \omega_2 \wedge \omega_2 \ldots + a_3 \omega_2 \wedge \omega_3 \ldots = 0$. ☐

Note that the above theorem implies that, given any $\omega \in \Omega^k$, it is a linear combination of $(v_1, \ldots, v_k) \rightarrow$ determinant of a $(k \times k)$ minor. Before going on to manifolds, we prove one more result.

Theorem 2.3. *Suppose $v_1, \ldots, v_n$ is a basis for V and $\omega \in \Omega^n$. Let $w_i = \alpha_i^j v_j$. Then $\omega(w_1, \ldots, w_n) = \det(\alpha_i^j)\omega(v_1, \ldots, v_n)$.*

Proof.

$$\omega(w_1,\ldots,w_n) = \omega(\alpha_1^{j_1}v_{j_1},\alpha_2^{j_2}v_{j_2},\ldots) = \alpha_1^{j_1}\alpha_2^{j_2}\ldots\omega(v_{j_1},\ldots)$$
$$= \alpha_1^{j_1}\alpha_2^{j_2}\ldots sgn(\sigma_j)\omega(v_1,\ldots,v_k), \tag{2.1}$$

where $\sigma_j(i) = j_i$. Therefore the theorem holds.
Alternatively, define $\eta((a_{11},\ldots,a_n1),(a_{21}\ldots),\ldots) = \omega(w_1,w_2,\ldots) = c\det$ where $c = \omega(v_1,\ldots,v_n)$. □

3. A digression - Orientation

In our study of the Möbius strip (in the HW) we found that it is not trivial and that is because of the intermediate value theorem. The real problem is that the Möbius strip has no outside or inside (if it did, it would have to be trivial). Moreover, even in the Green, div, and curl theorems, one needs to have a notion of an "outward pointing" normal. So, we need to make sense of the concept of "orientation".

Suppose V is a f.d. vector space. An orientated basis is simply an ordered basis $e_1,\ldots,e_n$. Two orientated bases are said to "agree" if the change of basis $\tilde{e}_i = a_i^j e_j$ satisfies $\det(a_i^j) > 0$. This defines an equivalence relation among orientated bases with exactly two equivalence classes called "orientations". An orientation preserving linear isomorphism $T : (V,\mu = [e_1,\ldots,e_v]) \to (W,\nu = [f_1,\ldots,f_v])$ satisfies $[f_1,f_2,\ldots] = [T(e_1),T(e_2),\ldots]$.

On $\mathbb{R}^n$, there is a "standard" orientation given by the standard basis. Note that in $\mathbb{R}^3$, the right-hand thumb rule gives orientations. Here are a couple of examples :

(1) $\vec{x} \to -\vec{x}$. This is orientation preserving on $\mathbb{R}^{2n}$ but not on $\mathbb{R}^{2n+1}$.
(2) $\vec{x} \to (x_2,x_1,x_3,\ldots)$. This is always orientation reversing.

we have the following useful lemma that follows from earlier theory easily.

Lemma 3.1. *If V is an n-dimensional vector space, and $0 \neq \omega \in \Omega^n$, then there is a unique orientation μ on V such that $[v_1,\ldots,v_n] = \mu$ if and only if $\omega(v_1,\ldots,v_n) > 0$. Moreover, every orientation arises this way.*

In other words, specifying an orientation on a vector space is the same as specifying a non-zero "top" order alternating tensor.

Suppose V is a smooth vector bundle over M. We want to make sense of when the vector bundle is orientable and give an orientation. Naively, we want to say that an orientation is simply a collection of orientations μ_p, one for each point $p \in M$. But somehow these orientations should "vary" smoothly. To make sense of this, consider a trivial bundle $U \times \mathbb{R}^r$ where $U \subset M$ is a connected open subset. Give every fibre the standard $\mathbb{R}^r$ orientation. Suppose $T : U \times \mathbb{R}^r \to U \times \mathbb{R}^r$ is a smooth bundle isomorphism, then T is either orientation-preserving or orientation-reversing on all fibres because $T_x(e_i) = a_i^j(x)e_j$ where $\det(a_i^j)(x) \neq 0 \; \forall \; x \in U$. Therefore the following definition makes sense.

Definition 3.2. A vector bundle V over a manifold M is said to be orientable if there is a choice of $\mu_p \; \forall \; p \in M$ such that every trivialisation $\Phi_U : \pi^{-1}(U) \simeq U \times \mathbb{R}^r$ over a connected open set U satisfies the following property : $\Phi_U(p)$ either preserves μ_p for all $p \in U$ or reverses μ_p for all $p \in U$ (where $\mathbb{R}^n$ is equipped with the standard orientation). A given choice of such a smoothly varying $\mu_p \; \forall \; p \in M$ is called an orientation.

Note that if one trivialisation t satisfies this property for U, then all trivialisations $t' : \pi^{-1}(U) \simeq U \times \mathbb{R}^r$ satisfy this property.

Definition 3.3. If (V, μ), (W, ν) are oriented vector bundles over M, then a smooth bundle isomorphism $T : V \to W$ is said to preserve the orientations if $T(\mu_p) = \nu_p \ \forall \ p \in M$. So we have an equivalence relation among the orientations on an oriented vector bundle. It is easy to see that there are only two equivalence classes.

As a consequence, suppose we cover M with trivialising open sets U_α. Assume that V is orientable, with a given choice of an orientation. Then after composing with one of the example maps described above, we can assume that $\Phi_U(p)$ preserves μ_p for all $p \in U$. Therefore $g_{\alpha\beta} = \Phi_\alpha \circ \Phi_\beta^{-1}$ has positive determinant for all α, β. Conversely, if the transition functions satisfy this property, then it is easy to see that the standard orientation induced from $\mathbb{R}^r$ defines an orientation on V. So an oriented vector bundle can be alternatively defined as
Alternative equivalent definition of orientability : A (real) vector bundle whose transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ can be chosen to have positive determinant, i.e., for all $\alpha, \beta, p \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(p) \in GL^+$ is said to be orientable.