NOTES FOR 15 SEPT (FRIDAY)

1. Recap

(1) Motivated the study of continuous symmetries from the perspective of differential equations.
(2) Defined Lie groups and Lie group homomorphisms. Proved that an embedded submanifold of a Lie group that is closed under multiplication and inversion is a Lie group in its own right.
(3) Gave examples (Abelian and Matrix examples).

2. Lie groups and Lie algebras

(1) We proved the symplectic group $Sp(2n, \mathbb{R})$ consisting of $2n \times 2n$ invertible matrices A such that $A^T J A = J$ is a Lie group.
Just as the orthogonal group consists of matrices that preserve a metric, $Sp(2n)$ preserves a so-called symplectic form, i.e., a non-degenerate bilinear form $\omega : V \times V \to \mathbb{R}$ satisfying $\omega(X, Y) = -\omega(Y, X)$. Every such form can be written (using an appropriate basis) to J. Indeed, we induct on n. For $n = 2$, take any e_1 and define e_2 as any linearly independent vector such that $\omega(e_1, e_2) = 1$. Suppose this has been done for $n = k$, then for $n = k + 1$, take e_1, e_2 such that $\omega(e_1, e_2) = 1$. Consider the $2k$-dimensional subspace S consisting of v such that $\omega(v, e_1) = \omega(v, e_2) = 0$. (This is $2k$-dimensional by non-degeneracy of ω.) Use the induction hypothesis. (Indeed, this proof shows that symplectic forms can exist only in even dimensions.) The point of studying symplectic forms is to generalised classical mechanics as formulated by Hamilton.
(2) $SU(n)$ and $SO(n)$ are also a Lie groups. Indeed, $SO(n)$ is a connected component of $O(n)$. As for $SU(n)$, consider the smooth map $\det : U(n) \to S^1$. We shall prove that 1 is a regular value, i.e., given any real number r, there is a curve of unitary matrices $U(t)$ passing through a given matrix U_0 such that $\frac{d\det(U(t))}{dt}|_{t=0} = \sqrt{-1}r$. Indeed, $U_0 = P^\dagger D P$ where $D = diag(e^{\sqrt{-1}a_1}, e^{\sqrt{-1}a_2}, \ldots)$. So define $U(t) = P^\dagger D(t) P$ where $D(t)$ is chosen so that $tr(D' D^{-1}) = \frac{\det(U(t))}{dt}|_{t=0} = \sqrt{-1}r$.
Another beautiful fact is that $SU(2) \equiv S^3$. Indeed, $SU(2)$ consists of matrices of the form $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ where $|a|^2 + |b|^2 = 1$. The smooth map $(a, b) \in S^3 \subset \mathbb{R}^4 \to SU(2)$ is a diffeomorphism. (Why ?)

The notion of a Lie subgroup is somewhat subtle. Naively one would think that a Lie subgroup is a subgroup which is also a submanifold. Unfortunately, that expectation is not quite correct because it can be an *immersed* submanifold (as opposed to an embedded one). Indeed, the immersion $f : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ given by $f(t) = (t, ct)$ where $c \notin \mathbb{Q}$ is an immersed but not an embedded submanifold because its image is dense. That being said, an embedded submanifold H which is also a subgroup is a Lie group in its own right because the map $(x, y) \to xy^{-1}$ is smooth when considered as a map into G.

So a Lie subgroup H is defined to be a subgroup $H \subset G$ such that H is a Lie group for some topology and C^∞ structure, and the inclusion map is a $1-1$ immersion. It is a very non-trivial result that *closed* Lie subgroups are actually embedded submanifolds. (Cartan's theorem.)

Note that $SO(n)$ is a connected component of $O(n)$ containing the identity. This can be generalised to

Theorem 2.1. *If G is a Lie group, then the component K containing the identity $id \in G$ is a closed normal Lie subgroup.*

Proof. Components are closed and hence K is so. The map $(x, y) \to xy^{-1}$ is continuous and hence sends $K \times K$ to a connected set containing id and hence to K. Thus K is a subgroup. Consider the continuous maps $a \to bab^{-1}$ from K to G (for every $b \in G$). Once again the image contains id and hence is contained in K for every $b \in G$. Thus K is normal. Since components are open, K is also an embedded submanifold. Thus K is a closed normal Lie subgroup. □

As a final example, let us consider the group of isometries $E(n)$ of $\mathbb{R}^n$, i.e., diffeomorphisms that preserve distances. It turns out that every such isometry is an affine map : $\vec{x} \to A(\vec{x} + \vec{a}) = A \circ \tau_a$ where τ_a is a translation and $A \in O(n)$. As a manifold $E(n)$ can be given the smooth structure of $O(n) \times \mathbb{R}^n$ (at least a set it is bijective to the latter). It is also a Lie group because $A \circ \tau_a \circ B \circ \tau_b(x) = A\tau_a(Bx + Bb) = A(Bx + Bb + a) = ABx + ABb + Aa$ (and $AB \in O(n)$) and $\tau_a^{-1} \circ A^{-1}(x) = A^{-1}x - a$ (and $A^{-1} \in O(n)$).
However, as a Lie group, it is *not* a direct product. Indeed, $A\tau_a A^{-1}(x) = A(A^{-1}x + a) = x + Aa \neq x + a$. It turns out that it is a semidirect product. It is easy to see that the component of identity is simply the above with $SO(n)$ matrices.

There are two maps (left and right translations) on a Lie group. $L_a(b) = ab$ and $R_a(b) = ba$. Since they are smooth and have smooth inverses $L_{a^{-1}}, R_{a^{-1}}$ we see that they are diffeomorphisms. Therefore it makes sense to pushforward vector fields. (Recall that if ϕ is a diffeomorphism, $(\phi_* X)(p) = \phi_*(X(\phi^{-1}(p)))$.)

A vector field X is said to be left invariant (resp. right invariant) if $L_{a*}X = X \ \forall \ a \in G$ (resp. $R_{a*}X = X \ \forall \ a \in G$). Suppose $X_a \in T_aG$ is the value of X at a, then for a left-invariant vector field, $L_{a*}X_b = X_{ab} \ \forall \ a \in G$ which means that in particular, $L_{a*}X_e = X_a \ \forall \ a \in G$. Moreover, this is equivalent to left-invariance. (Why?) Therefore, given an element $X_e \in TG_e$ we can produce a unique left-invariant vector field X such that $X(e) = X_e$ (and all left-invariant vector fields arise this way). But it is not obvious yet that all left-invariant vector fields are smooth.

Proposition 2.2. *Every left-invariant vector field (simply a left-invariant section of TG) is smooth.*

Proof. By the above discussion, $X(a) = L_{a*}X_e$. Suppose (x, U) is a coordinate system near e. Then $y = L_a(x), L_a(U)$ is a coordinate system near a (because L_a is a diffeomorphism). Now suppose $b \in L_a(U)$. Then $X(b) = L_{b*}X_e$ in coordinates is $(y^1, \ldots, y^m) \to \frac{\partial y^i \circ L_{b(y)}}{\partial x^j} X_e^j \frac{\partial}{\partial y^i}$ which is smooth in y because multiplication is smooth. □

An easy corollary is that every Lie group is parallelizable, i.e., it has trivial tangent bundle. (Just take a basis for T_eG and construct left-invariant vector fields.) Therefore S^3 is parallelizable. Moreover, S^2 is therefore *not* a Lie group. Actually it turns out that other than S^1 and S^3 no odd sphere is a Lie group.