NOTES FOR 27 OCT (FRIDAY)

1. Recap

(1) Wrote a basis for Ω^k. Prove that a bunch of 1-forms are independent if and only if their wedge product is non-zero. Also proved that for top order forms, if you change a basis, then you pick up a factor of the determinant.
(2) Defined orientation of vector spaces and orientation-preserving maps. Proved that this is the same as providing a top-order alternating tensor.
(3) Defined orientation for vector bundles. Proved that it is equivalent to being able to trivialise using transition functions that have positive determinant.

2. A digression - Orientation

If $V = TM$, then

Definition 2.1. A manifold M is said to be orientable if TM is orientable. Equivalently, if there exists a choice of connected coordinate charts (x_α, U_α) covering M so that $\det(\frac{x^i_\alpha}{x^j_\beta}) > 0 \;\forall\; \alpha, \beta$, then M is said to be orientable with the orientation given by the charts. A diffeomorphism $f : M \to N$ between oriented manifolds is said to be orientation preserving (reversing) if $f_* : T_pM \to T_{f(p)}N$ is orientation preserving (reversing) for all p. Equivalently, if we choose charts compatible with the given orientations, then $det(Df) > 0$.

Here are examples and counterexamples :

(1) The trivial bundle is orientable. Hence the n-torus $S^1 \times S^1 \ldots$ is an orientable manifold.
(2) We will prove later on that the sphere S^n is orientable. More generally, any hypersurface $f^{-1}(0) \subset \mathbb{R}^{n+1}$ is orientable if $\nabla f(p) \neq 0 \;\forall\; p \in f^{-1}(0)$. (Essentially, the orientation is given by either the outward or inward pointing normal N, i.e., a basis $e_1, \ldots, e_n$ in the T_pS is an orientation if $[N(p), e_1, \ldots, e_n]$ is the standard orientation in $\mathbb{R}^{n+1}$. Proving this varies smoothly is not too hard but it is cleaner to do it later using differential forms.)
(3) The Möbius line bundle over S^1 is not orientable. Indeed, if it had an orientation μ, then define a nowhere vanishing smooth section s as $s(p) \in \mu_p \cap \{1, -1\}$. It is easy to see (using coordinates) that this varies smoothly.
(4) The open infinite Möbius strip as a manifold is not orientable. Note that the central circle is orientable using the vector field $X = \frac{\partial}{\partial \theta}$. So, define an orientation ν on the Möbius line bundle by saying that $v_p \in \nu_p \Leftrightarrow [v_p, X(p)] \in \mu_p$. It is easy to see (as above) that it varies smoothly.
(5) $\mathbb{RP}^2$ is not orientable because it contains the Möbius strip (so any compatible set of coordinate charts would have covered the Möbius strip too but we know that cannot happen). (In fact, this is the case for all even dimensional real projective spaces but a clean proof of this can be given using differential forms which we will do later.)
(6) $\mathbb{RP}^3$ is orientable. Use the orientation inherited from S^3 via the antipodal map $p \to [p]$. This is valid because I claim that this map, when written in coordinates, has a positive determinant for its derivative. (Similar for odd real projective spaces.)

3. Differential forms on manifolds

Suppose M is a manifold. Then, we can form the bundle $\Omega^k(M)$ by first considering it as a set $\Omega^k(M) = \cup_{p\in M}\Omega^k_p = T^*_pM \wedge T^*_pM \ldots$, i.e., at every point, take the rank-k alternating tensor space. As usual, we give it a topology and a vector bundle structure over M by taking coordinate charts (x, U) on M, and trivialising $\Omega^k(U)$ by using $dx^{i_1} \wedge dx^{i_2} \ldots$ (where $1 \leq i_1 < i_2 \ldots$). A smooth section ω of $\Omega^k(M)$ is called a differential form (field). Locally, it is of the form, $\omega = \omega_{i_1 i_2 \ldots}(x) dx^{i_1} \wedge dx^{i_2} \ldots = \omega_I(x) dx^I$ where $\omega_I(x)$ are smooth functions. If you change coordinates to $(\tilde{x}, U)$, then the components change as $\tilde{\omega}_I = \omega_J k! Alt_I(\frac{\partial x^{j_1}}{\partial \tilde{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \tilde{x}^{i_2}}) \ldots$ where Alt_I is antisymmetrization on the I-indices, i.e., $k!Alt_I(A^{j_1}_{i_1} A^{j_2}_{i_2} \ldots)$ is the determinant of the $k \times k$ minor of the matrix A^j_i consisting of the J rows. In fact, this construction can be generalised to general vector bundles V. Indeed, if the transition functions of V are $g_{\alpha\beta}$, then those of $\Omega^k(V)$ are $(h_{\alpha\beta})^J_I = k!Alt_I([g_{\alpha\beta}]^{j_1}_{i_1}[g_{\alpha\beta}]^{j_2}_{i_2} \ldots)$.

Given a k-form field ω on N and a smooth map $f : M \to N$, we can define the pullback $f^*\omega$ as a smooth k-form field on M. Indeed, $(f^*\omega)_p(v_1, v_2, \ldots, v_m) = \omega_{f(p)}(f_*v_1, f_*v_2, \ldots)$. Before proceeding to find out its expression in coordinates, we observe that we can define the wedge product of differential form (fields) $\omega \wedge \eta$ as $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$. This produces a smooth $k + l$-form. Indeed, if $\omega = \omega_I dx^I$ and $\eta = \eta_J dx^J$, then $\omega \wedge \eta = \omega_I \eta_J dx^I \wedge dx^J$ which is smooth (because the components are quadratic polynomials in smooth functions). An example of a wedge product in $\mathbb{R}^5$: $\omega = x^1 dx^2 \wedge dx^3 + e^{x^2} dx^1 \wedge dx^5$ and $\eta = 3dx^1 \wedge dx^4$. Now $\omega \wedge \eta = 3x^1 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$. The wedge product satisfies the usual properties :

1. It is bilinear over the ring of smooth functions.
2. $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$,
3. $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Now we evaluate the coordinate expression of the pullback.

Theorem 3.1. *If $f : M \to N$ is a smooth function, (x, U) is a coordinate system near $p \in M$ and (y, V) near $f(p) \in N$, then*

$$f^*(g dy^1 \wedge \ldots dy^k) = (g \circ f) \det(\frac{\partial y^\alpha \circ f}{\partial x^{j_\beta}}) dx^{j_1} \wedge \ldots dx^{j_k} \tag{3.1}$$

where as before, the sum over multi-indices is always over increasing indices, i.e., $1 \leq j_1 < j_2 \ldots$.

Proof.

$$(f^*(g dy^1 \wedge \ldots dy^k))_x(\frac{\partial}{\partial x^{j_1}}, \frac{\partial}{\partial x^{j_2}}, \ldots) = (g dy^1 \wedge \ldots dy^k)_{f(x)}(f_* \frac{\partial}{\partial x^{j_1}}, f_* \frac{\partial}{\partial x^{j_2}}, \ldots)$$
$$= g(f(x)) dy^1 \wedge \ldots dy^k (\frac{\partial y^a \circ f}{\partial x^{j_1}} \frac{\partial}{\partial y^a}, \ldots)$$
$$= g(f(x)) \frac{\partial y^a \circ f}{\partial x^{j_1}} \frac{\partial y^b \circ f}{\partial x^{j_2}} \ldots dy^1 \wedge \ldots dy^k (\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}, \ldots) = (g \circ f) \det(\frac{\partial y^\alpha \circ f}{\partial x^{j_\beta}}) \tag{3.2}$$

∎