NOTES FOR 28 AUG (MONDAY)

1. Recap

(1) Defined the cotangent bundle and vector bundles. Proved that the cotangent bundle is a vector bundle.
(2) Defined sections and proved that $df : M \to T^*M$ is a section.
(3) Proved that dx^i are local sections that span the fibres at every point and hence every section of T^*M (1-forms) is locally a smooth linear combination of dx^i.

2. Vector fields, Tangent bundle, Cotangent bundle, etc

Given a smooth map $f : M \to N$, a smooth 1-form η on N, there is a way to construct a smooth 1-form on M denoted as $f^*\eta$ and called the "pullback" of η. Indeed, suppose locally we choose coordinates $(x, U), (y, V)$ on M and N respectively, and $\eta(q) = [q, \eta_i(q) dy^i(q)]$ where $\eta_i : V \subset N \to\to T^*N$ are smooth functions transforming "correctly" under change of coordinates. Now define $f^*\eta(p) = [p, \eta_i \circ f(p) \frac{\partial f^i}{\partial x^j}(p) dx^j(p)]$. Verify that indeed this is a valid smooth 1-form (i.e., does it change as a one-form is supposed to under changes of coordinates ?).

Returning back to vector bundles, we need to decide when two vector bundles are to be considered the same (in some sense):

Definition 2.1. A *bundle map* (or sometimes *bundle morphism*) from E_1 over M_1 to E_2 over M_2 is a pair of smooth maps $f : M_1 \to M_2$, $\tilde{f} : E_1 \to E_2$ such that $f \circ \pi_1 = \pi_2 \circ \tilde{f}$ and $\tilde{f} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is a linear map between vector spaces.

If $M_1 = M_2$, $f = Identity$, and $\tilde{f}$ is a diffeomorphism that is an isomorphism of vector spaces for every p, then $\tilde{f}$ is said to be an *isomorphism* between the bundles E_1 and E_2. E_1 and E_2 are then said to be *isomorphic*.

Remark 2.2. It turns out that every vector bundle as defined above is isomorphic to one that has a weird construction similar to the one we gave for T^*M using a quotient of a disjoint union.

Here are two examples of vector bundles :

(1) The cotangent bundle T^*M of a manifold M is a rank-m vector bundle. An example of a section of T^*M is df where $f : M \to \mathbb{R}$ is a smooth function.
(2) The trivial rank-r vector bundle $T = M \times \mathbb{R}^r$. All of its smooth sections are of the form $s : M \to T$ given by $s(p) = (p, f(p))$ where $f : M \to \mathbb{R}^r$ is a smooth function.

We will construct more examples as we go along. (Spoiler alert : The next example will be the tangent bundle.)

Our next "natural question" will have to do with physics and engineering : Suppose you want to model the flow of rain water on the surface of our IISc roads. Of course the roads are not smooth and water can seep into them. But let's ignore such real world complications. Effectively, what we are saying is that we have a submanifold $Road \subset \mathbb{R}^3$ and a vector-valued function $\vec{X} : Road \to \mathbb{R}^3$ that tells you the velocity of the water at every point on the road. Of course has to be a smooth function. More importantly, every such vector has to be "tangent" to the road, i.e., if locally, the road looks like $f^{-1}(0)$ for some function f, then $\nabla f.\vec{X} = 0$ (the vectors should be perpendicular to the normal). Alternatively, at every point $p \in Road$, there exists a smooth curve $\vec{c} : (-\epsilon, \epsilon) \to Road$ such that $c^{'}(0) = \vec{X}(p)$ where $\vec{c}(0) = p$. Such a vector-valued function is called "a smooth tangent vector field" or sometimes, simply, "a vector field". If we now want to study hydrodynamics on a general manifold M, one possible way is to embed it in $\mathbb{R}^N$ using the Whitney theorem, and define vector fields as above. But we want to do this without any reference to $\mathbb{R}^N$, i.e., intrinsically. So how may we define a "tangent vector space" T_pM at every point p ? Also, how can we make the notion of "smoothly varying tangent vectors, i.e., a vector field" precise as perhaps a smooth map $X : M \to TM$ where TM is some other manifold ?

We could try to define a smooth vector field as a collection of smooth vector-valued functions $\vec{X}_\alpha : U_\alpha \to \mathbb{R}^n$ (where U_α forms an atlas) such that when you change coordinates, the vector field transforms in a particular manner and try to construct TM akin to T^*M. But we will construct in a more abstract manner by first defining the "tangent spaces" T_pM and making $TM = \coprod_{p\in M} T_pM$ into a manifold. This is just for the sake of variety more than anything else.

There are two ways to define tangent spaces. One of them is concrete and very natural. The other is abstract but is useful in some contexts.

(1) T_pM as an equivalence class of smooth curves passing through p : Let C be the set of all smooth curves $c : (-1, 1) \to M$ such that $c(0) = p$. Define a relation $\sim$ on C as $c_1 \sim c_2$ if for every coordinate chart $\Phi : U \to \mathbb{R}^m$ where $p \in U$, $\frac{d(\Phi\circ c_1)}{dt}|_{t=0} = \frac{d(\Phi\circ c_2)}{dt}|_{t=0}$. Denote the equivalence class of c as $[c]$ and define T_pM as the set of equivalence classes $\{[c] | c \in C\}$. I claim that this is an m-dimensional vector space.

Firstly, to check for equivalence, it is enough to do so in one coordinate chart (why?). Fix a coordinate chart $(U, x = \Phi)$ around p such that p is at the origin of this coordinate system. Given a vector $v \in \mathbb{R}^m$, define a curve $\tilde{c}_v : (-1, 1) \to \mathbb{R}^m$ as $\tilde{c}_v(t) = tv$. This curve passes through the origin and has a tangent vector v there, i.e., $\tilde{c}_v^{'}(0) = v$. Now define a map $h : \mathbb{R}^m \to T_pM$ as $h(v) = [\Phi^{-1} \circ \tilde{c}_v]$. I claim that this is a bijection (thus proving that T_pM has a vector space structure).

(a) h is injective : If $h(v_1) = h(v_2)$, then $\frac{d\tilde{c}_{v_1}}{dt}(0) = v_1 = \frac{d\tilde{c}_{v_2}}{dt}(0) = v_2$.

(b) h is surjective : Suppose we take $[c] \in T_pM$ and consider a representative $c \in C$. Then $\frac{d(\Phi\circ c)}{dt}(0) = v \in \mathbb{R}^m$. Now $[h(v)] = [c]$ (why ?)

(2) T_pM as point-derivations on the algebra of germs of smooth functions at p : To be continued....