NOTES FOR 2 AUG (WEDNESDAY)

1. Logistics

(1) Welcome to 338 - Differentiable manifolds and Lie Groups. We may or may not deal with Lie groups (despite the title). I will mostly follow Spivak's Comprehensive Introduction to Differential Geometry (Vol 1). Other references are there on the webpage. I will more less follow the syllabus mentioned in http://www.math.iisc.ac.in/courses/aug_dec2014/Differentiable_Manifolds_and_Lie_Groups.htm. However, we might go over vector bundles instead of homogeneous spaces.
(2) As for the grading scheme, there will be one midterm (carrying 30% of the grade), HW (20%), and a final (50%). Please note that the HW are expected to be written clearly and rigorously on stapled pieces of paper. I would strongly encourage writing them in LaTeX. But that is completely upto you.
(3) The course webpage is http://math.iisc.ac.in/~vamsipingali/338_2017.html. It contains the HW, the dates of the exams, the PDFs of the notes, the lecture schedule, etc. Please visit it frequently.
(4) My email address is *vamsipingali@math.iisc.ac.in*. My office is N-23. I will hold "office hours" on Wednesdays from 3:00-4:00 in my office whence you can ask me questions regarding this course. (Of course you can always email me as well. This is just to maintain a routine.)

2. Motivation and history (liberally copied from Wikipedia)

(1) The ancient civilisations (Indus-Valley, Babylonian, Egyptian, and Chinese) all knew some geometric figures (like cylinders) and a little bit of measurement (after all they knew architecture to some extent). There seems to be evidence that the Babylonians knew the Pythagoras theorem much before the Vedic Indians and hence Pythagoras himself.
(2) The first milestone comes with Euclid in the 3rd century BC. He axiomatised geometry in the form of five axioms. (Currently, the most popular modern version of them are the Hilbert's axioms.) His fifth axiom was "If a transversal cuts two lines such that the sum of the interior angles is less than two right angles, then these two lines intersect." A famous question of that time was "Can this postulate be derived from the other 4?"
(3) Along different lines, during the Renaissance (14th century), the concept of "perspective art" came into prominence. Perspective deals with drawing three-dimensional objects on a piece of paper as the eye would see them. This lead to a different facet of geometry - One that de-emphasised distances and angles, and instead focussed on points of intersection of lines. This is called projective geometry and is instrumental in the development of modern algebraic geometry.
(4) In the 17th century, two important developments took place - Descartes and Fermat introduced coordinates, thus opening up analytic/coordinate geometry. Whatever could be done earlier by clever geometric arguments became routine algebra. The other important discovery was calculus by Newton and Leibniz. Their work answered two questions thought to

be intractable - finding tangents to general curves (cubics for instance) and the area under curves. (Special cases were studied in great detail by Indians much earlier.)

(5) Another important development in the 17th century was the solution of the seven bridges problem by Euler. He discovered graph theory as well as the formula $V - E + F = 2$ thus beginning the study of topology.

(6) In the 19th century, the old problem of Euclid's fifth postulate resurfaced. Gauss, Bolyai, and Lobatchewsky constructed a geometry (hyperbolic geometry) which was more or less consistent, but violated the fifth postulate. This was the birth of non-Euclidean geometries. Klein constructed a "coordinate" model of hyerbolic geometry.

(7) During the same time, Gauss answered an important cartographical question - "Can one draw the map of let's say a part of Germany on a piece of paper without distorting distances ?" (The usual Mercator projection we are used to seeing in Atlases preserves angles but distorts distances heavily) The answer (in the form of Theorema Egregium) that Gauss provided was NO. To do this, he calculated a quantity called "Gaussian curvature" and proved that it does not depend on whether you put Germany on a piece of paper or on the earth. But earth is curved whereas the paper is not.

(8) Gauss's student Riemann began a systematic study of higher dimensional versions of surfaces. His main point was to use coordinates to parametrise objects (just like latitude and longitude on the earth, the positions and momenta of all the particles in a room, etc). He, like Gauss, defined quantitites that "behaved well" under changes of coordinates.

(9) In the late 19th and early 20th centuries, Poincare developed Algebraic topology (fundamental group, homology, etc) in his famous Analysis Situs papers.

(10) Albert Einstein used Riemannian geometry in his General theory of relativity and motivated further development.

(11) Whitney and Whitehead defined manifolds rigorously and clarified the foundations of the subject.

(12) In 2006, Perelman proved the Poincare conjecture.

In simple terms, given two intersecting straight lines, the question "What is the angle made by these lines ?" has to do with Differential geometry, "How many points do these lines intersect in?" has to do with Algebraic geometry, and "If the lines are made of rubber, can you deform them to a single line ?" is a topological question.

3. Definition of a manifold and examples

The above is enough motivation for the definition of a manifold but here is a problem that will serve as a guide for our definitions :
Suppose $P : S^2 \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function. Where does it achieve its maximum ?
Firstly, what does it mean for P to be differentiable ? (After all, S^2 is a closed set.) It means that P is differentiable on an open set containing S^2. Secondly, the answer is that the maximum is achieved at a place where $\frac{\partial P}{\partial \theta} = \frac{\partial P}{\partial \phi} = 0$ where θ, ϕ are the latitude and longitude respectively. (Why?) I claim that this condition does not depend on what way one parametrises the sphere, i.e., if you choose (x, y) coordinates where $z = \pm\sqrt{1 - x^2 - y^2}$, you will get the same answer. Indeed, the chain rule of multivariable calculus guarantees this.

The above problem and the aforementioned history shows that we need to define "sphere-like" objects (manifolds) in such a way that

(1) A manifold can be parametrised (at least locally) by coordinates, i.e., it locally looks like $\mathbb{R}^n$.
(2) If a function is differentiable in one coordinate system, it is so, in all the others.
(3) Gauss' theorema egregium shows that we must focus our efforts into defining quantities that "behave well" under coordinate changes.
(4) One of the aims of studying manifolds ought to be to "classify" them, i.e., "write" a "standard list" of them such that every manifold is one element of that hypothetical list.

Keeping these requirements in mind, here is the definition of a *Topological manifold* (as opposed to a *smooth manifold* which will be our main object of study later on) :

Definition 3.1. A topological manifold M is a Hausdorff, second countable space that is locally Euclidean, i.e., it is covered by open sets U_α such that there exist homeomorphisms $\Phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{n_\alpha}$ where $n_\alpha \geq 0$ is an integer and V_α is an open subset of $\mathbb{R}^{n_\alpha}$.

Remark 3.2. By the way, if instead of being "modelled" by R^{n_α} we use an infinite dimensional Hilbert or a Banach space, then we will get an infinite dimensional Hilbert or Banach manifold. Such seemingly abstract objects are actually quite useful in the study of (nonlinear) PDEs.

Remark 3.3. It is obvious that the n_α need not all be the same. For example, take $M = \mathbb{R}^2 \cup \mathbb{R}$. However, this is cheating because M is disconnected. For connected manifolds (and whenever I say manifold in this course, unless I specify otherwise, it is assumed to be connected) are all the n_α equal to each other ? The answer is YES. But we will discuss it in a moment.

Remark 3.4. Why Hausdorff (separating points by open sets) and second countable (countable basis) ? Hausdorff guarantees uniqueness of limits of sequences. Second countable (which is sometimes replaced by a weaker requirement - paracompact) is necessary for a more technical reason - partitions-of-unity. Ultimately these requirements stem from the desire of seeing every manifold arising as a subset of $\mathbb{R}^n$ (and hence as a metric space). A caveat - there are examples of non-Hausdorff second countable spaces that are locally Euclidean. An infamous example is the "double line" (see Hawking and Ellis' Large scale structure of spacetime for instance) - Take 2 copies of the real line and identify the corresponding strictly negative numbers on both lines.

As mentioned in remark 3.3, if M is connected (which we will assume without mention from now on), the n_α are all equal to n. This number is called the dimension of the manifold. (Sometimes M is called an "n-fold".) Indeed, this is a corollary of the following famous theorem. (Why is this fact a corollary ?)

Theorem 3.5. *Invariance of Domain*
If $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous injection from an open set U to its image, then the image of f is open in $\mathbb{R}^n$.

Remark 3.6. The proof of this theorem involves some complicated Algebraic topology. It follows from the generalised Jordan curve theorem (see Spivak). But the latter still involves complicated algebraic topology.

What are examples of topological manifolds ? Any open subset of $\mathbb{R}^n$ should do the job. We can easily come up with more examples (check that the circle is one) but let us defer that job for later discussion of smooth manifolds.

The above definition is unfortunately not good enough to state "optimisation" problems like our guiding one. Indeed, how can one define the notion of a differentiable function on a topological manifold ? One might be tempted to use the local coordinates, i.e., $P \circ \Phi_\alpha^{-1} : V_\alpha \subset \mathbb{R}^n \to \mathbb{R}$ is an ordinary function for which differentiability makes sense. The problem is that if we change coordinates, i.e., on $U_\alpha \cap U_\beta$, there is no guarantee that $P \circ \Phi_\beta^{-1} = P \circ \Phi_\alpha^{-1} \circ \Phi_\alpha \circ \Phi_\beta^{-1}$ is still differentiable because the "transition map" $\phi_{\alpha\beta} = \Phi_\alpha \circ \Phi_\beta^{-1}$ is not necessarily a differentiable map. So we need to remedy this through a new definition. Here is a naive definition of a differentiable manifold. We will replace this definition with the "correct" one in a moment.

Naive definition : A C^k manifold $(M, \{U_\alpha, \Phi_\alpha\})$ is a Hausdorff, second countable space M that is covered with open sets (called coordinate charts) U_α such that there exist homeomorphisms $\Phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ with the transition functions $\phi_{\alpha\beta} = \Phi_\alpha \cap \Phi_\beta^{-1} : \Phi_\beta(U_\alpha \cap U_\beta) \to \Phi_\alpha(U_\alpha \cap U_\beta)$ are C^k diffeomorphisms, i.e., there are k-times continuously differentiable with their inverses being k-times continuously differentiable. Such charts are called "C^k compatible". The collection of C^k compatible charts $\{U_\alpha, \Phi_\alpha\}$ covering M is called an *atlas* (because it is a collection of "charts"). If $k = 0$ it is a topological manifold. If $k = \infty$ then it is called a *smooth* manifold.

Remark 3.7. Instead of C^k, we can study C^ω which is the class of real analytic manifolds (the transition functions are locally power series). Likewise, we can have complex manifolds which locally look like $\mathbb{C}^n$ and the transition maps are biholomorphisms. The latter are quite useful in the study of algebraic geometry.

Why is this definition naive ? This is because, suppose we take an atlas, and add another open set U such that the new transition maps are still C^k diffeomorphisms, then according to our naive definition, we have obtained a "new" C^k manifold. But this is stupid. Clearly, it is the same beast. The same kind of calculus can be done on both (try solving an optimisation problem for instance). So such things should not give rise to new objects. To this end, we first make a definition and then prove a lemma.

Definition 3.8. A C^k atlas $\mathcal{U}$ is defined to be contained $\leq$ in a C^k atlas $\mathcal{V}$ if every chart in $\mathcal{U}$ is a chart in $\mathcal{V}$, i.e., if $(U, \Phi) \in \mathcal{U}$ then $(U, \Phi) \in \mathcal{V}$. A maximal C^k atlas is one that is not contained in any other C^k atlas.