NOTES FOR 30 AUG (WEDNESDAY)

1. Recap

(1) Defined the pullback of forms and bundle morphisms.
(2) Motivated the need for a tangent space (and perhaps the collection of tangent spaces to define vector fields).
(3) Defined the tangent space in one way using curves.

2. Vector fields, Tangent bundle, Cotangent bundle, etc

There are two ways to define tangent spaces. One of them is concrete and very natural. The other is abstract but is useful in some contexts.

(1) T_pM as an equivalence class of smooth curves passing through p.
(2) T_pM as point-derivations on the algebra of germs of smooth functions at p : Consider the set $\mathcal{F}_U$ of all smooth functions $f : p \in U \subset\rightarrow M \rightarrow \mathbb{R}$. Define an equivalence relation on $\mathcal{F}_U$ as $f \sim g$ iff there is a neighbourhood $p \in V \subset U$ such that $f|_V = g|_V$. The set $\mathcal{G}_{p,U}$ of equivalence classes is called "the Germs of smooth functions". This set is independent of U. Indeed, I claim that $\mathcal{G}_{p,U} = \mathcal{G}_{p,M}$ thus showing independence (and the germs will be denoted as $\mathcal{G}_p$). I leave the proof as an exercise (Hint: Use bump functions).

   Germs form an algebra under multiplication. (Why is this well-defined at the level of equivalence classes ?) Now we define the notion of a point-derivation : A point-derivation $D : \mathcal{G}_p \rightarrow \mathbb{R}$ is a linear map such that $D([fg]) = [f(p)Dg + g(p)Df]$ (check that this is well-defined). Define T_pM as the set of point-derivations. This set has an obvious vector space structure : $(\lambda D)(f) = \lambda D(f)$ and $(D_1 + D_2)(f) = D_1(f) + D_2(f)$ (check these obey the point-derivation property). We claim that this is isomorphic to $\mathbb{R}^m$.

   Indeed, firstly here is an example of a point-derivation : Choose a coordinate chart (x, U) around p. Then $D_i = \frac{\partial}{\partial x^i}(p)$ is a derivation. Indeed, $D_i f$ is well-defined at the level of germs and is linear (why?). The Leibniz rule shows that the point-derivation property is satisfied. Thus, there is a well-defined linear map map $h : \mathbb{R}^m \rightarrow T_pM$ satisfying $h(v) = v^i D_i = v^i \frac{\partial}{\partial x^i}$. We will show that it is an isomorphism :
   (a) h is injective : If $h(v) = 0$, then $h(v)(x^i) = [0]\ \forall\ i$. Thus $v^i = 0\ \forall\ i$.
   (b) h is surjective : Suppose $D \in \mathcal{G}_p$. Define $v^i = D(x^i)(p)$. I claim that $h(v) = D$. Indeed, $D(f - f(p)) = Df$ and so we may assume that $f(p) = 0$ without loss of generality. This means that $f(x) = g_i(x)x^i$ for some smooth functions $g_i(x)$ in a neighbourhood of p. (Indeed, this follows from Taylor's theorem.) Therefore, $Df = g_i(p)v^i = h(v)(f)$.

Given a smooth map $f : M \rightarrow N$, this induces a map $f_* : T_pM \rightarrow T_{f(p)}N$ (called the pushforward or differential of f) :

(1) As per the first definition, $f_*([c]) = [f \circ c]$.
(2) As per the second definition, $f_*(D)[g] = D[g \circ f]$.

Note that the second definition (actually, even the first) has one nice property : The point-derivations $D_i = \frac{\partial}{\partial x^i}(p)$ give an isomorphism between $\mathbb{R}^m$ and T_pM for all $p \in U_\alpha$. So, morally speaking, $D_i = \frac{\partial}{\partial x^i}$

should be thought of as vector fields on U_α such that at every point p, they form a basis for the tangent space T_pM. This means that, at least morally, every smooth vector field X on M should satisfy :

(1) $X(p)$ should lie in the tangent space T_pM.
(2) X should locally, in a coordinate chart U_α be $X = X^i(x)\frac{\partial}{\partial x^i}$ where $X^i(x) : U_\alpha \to \mathbb{R}$ are m smooth functions.

Indeed, we will define a topology such that $TM = \coprod_p T_pM$ is a vector bundle of rank-m, $D_i = \frac{\partial}{\partial x^i}$ give local sections, the local trivialisations are given by D_i, and vector fields are smooth sections of this bundle. Actually, all these requirements (coupled with some more natural ones) do the job :

Theorem 2.1. *To every manifold M, there corresponds a rank-m vector bundle $TM = \cup_{p\in M}T_pM$ (called the tangent bundle) such that for every smooth map $f : TM \to TN$ there exists a bundle map $f_* : TM \to TN$ satisfying :*

(1) *The map $\pi : TM \to M$ is $\pi(v_p \in T_pM) = p$, i.e., set theoretically, the map takes the tangent space at p to p.*
(2) *f_* when restricted to T_pM is the pushforward we defined earlier.*
(3) *$Id_* = Identity$. Moreover, if $g : N \to P$, then $(g\circ f)_* = g_* \circ f_*$.*
(4) *There is a bundle isomorphism $t_n : T\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n$ such that for every smooth function $f : \mathbb{R}^m \to \mathbb{R}^n$, $t_n \circ f_* : T\mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n = Df \circ t_m$ where $Df : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ is $(Df(v))^i = \frac{\partial f^i}{\partial x^j}v^j$.*

Such a correspondence is essentially unique : If there is another choice $T'M$ and another choice of map $f_\#$, then there are bundle isomorphisms $e_M : TM \to T'M$ such that $e_N \circ f_* = f_\# \circ e_M$.

Proof. The proofs are in Spivak. The uniqueness proof is particularly annoying and not at all useful to mammals. The existence proof is more illuminating and we will give a sketch : Firstly, as a set $TM = \coprod_{p\in M} T_pM$ equipped with a set-theoretic map $\pi : TM \to M$ given by $\pi(v \in T_pM) = p$. Therefore, $\pi^{-1}(p) = T_pM$. Given a smooth map $f : M \to N$, set-theoretically, we have a map $f_* : TM \to TN$ given by $f_*|_{T_pM}$ is the pushforward we defined earlier to $T_{f(p)}N. \subset TN$. It is easy to verify that Id_* is identity and $(f\circ g)_* = f_* \circ g_*$ at the level of tangent spaces. Now all we have to do is to put a topology on TM that makes it a vector bundle such that π, f_* are smooth maps with π satisfying the local triviality property and f_* being a bundle map.

Indeed, we shall define the topology by giving a basis of coordinate open sets. Recall that if $(\Phi = x, U)$ is a coordinate chart on M, then $e_i(p) = \frac{\partial}{\partial x^i}(p)$ form a basis for T_pM (when interpreted as point-derivations). Note that e_i are set-theoretically speaking, sections of TM, i.e., $\pi \circ e_i(p) = p$. Consider the map $h : \Phi(U) \subset \mathbb{R}^m \times \mathbb{R}^m \to TM$ given by $h(x, v) = v^i e_i(\Phi^{-1}(x))$. By definition of a basis, this is indeed a bijection. Define a topology by considering $h(U \times \mathbb{R}^m)$ as a basis for all coordinate charts U. This automatically makes TM locally euclidean with smooth transition functions (why?(and π a smooth map (why?). It is easy to check Hausdorffness and paracompactness using π (why?) So TM by construction is a vector bundle over M. Now we shall check that f_* is smooth. Indeed, suppose we choose coordinate charts $(x, p \in U)$ on M and $(y, f(p) \in V)$ on N, then $f_*(e_i)(y^j) = e_i(f^j) = \frac{\partial f^j}{\partial x^i}$. Thus, $f_*(v) = v^i \frac{\partial f^j}{\partial x^i}\frac{\partial}{\partial y^j}$ and hence f_* is smooth (and at the same time, the last property of the theorem is satisfied). □