NOTES FOR 30 OCT (MONDAY)

1. Recap

(1) Defined orientations on manifolds. Gave several examples of orientable and non-orientable manifolds. (The Möbius strip and even projective spaces are non-orientable whereas odd projective spaces and hypersurfaces of $\mathbb{R}^{n+1}$ which have a unit normal vector field are orientable.)
(2) Defined the bundle of differential forms on manifolds (and generalised it to exterior powers of vector bundles). Saw how smooth forms changed under change of coordinates (involved some ugly determinants of minors of the derivative maps), defined wedge product, pullback and computing the pullback in coordinates.

2. Differential forms on manifolds

By the way, recall for the record that if ω is a top form, i.e., $\omega = gdx^1 \wedge \ldots dx^m$, then if you change coordinates to $\tilde{x}$, then $\tilde{g} = g\det(\frac{\partial x^i}{\partial \tilde{x}^j})$. From now onwards, let's make a small change in the summation over mult-indices. Let us sum over all multi-indices and divide by $k!$, i.e., write $\omega = 2dx^1 \wedge dx^2 = dx^1 \wedge dx^2 - dx^2 \wedge dx^1$. We will find this way of doing things more convenient.

Returning to the concept of an orientation, note that if we have a nowhere zero top form $\omega \in \Omega^m(M)$, then at every point p, ω_p induces an orientation μ_p on T_pM. I claim that μ_p varies smoothly, i.e., it is locally standard. Indeed, suppose (x, U) is a connected coordinate chart around p, then $[\frac{\partial}{\partial x^1}(q), \ldots] = \mu_q \Leftrightarrow \omega_q(\frac{\partial}{\partial x^1}(q), \ldots) > 0$. But if this is the case at p, then this will definitely be the case on all of U by smoothness of ω and connectedness of U. Thus ω defines an orientation on M. Conversely,

Theorem 2.1. *An orientable manifold M admits a nowhere zero smooth m-form ω.*

Proof. We have shown that orientability is equivalent to covering by coordinate charts such that $\det(\frac{\partial \vec{x}_\alpha}{\partial \vec{x}_\beta}) > 0$. On $(\vec{x}_\alpha, U_\alpha)$ define an m-form form $\omega_\alpha = dx^1_\alpha \wedge dx^2_\alpha \ldots dx^m_\alpha$. Note that ω_α is nowhere vanishing on U_α. Using paracompactness we may assume that U_α is locally finite without loss of generality. Now take a partition-of-unity ρ_α subordinate to U_α (indexed by the same set. So ρ_α need not have compact support). Define $\omega = \sum_\alpha \rho_\alpha \omega_\alpha$. This I claim is nowhere 0. Indeed, at p, we sum over the finitely many (but non-empty) α_i such that $p \in U_{\alpha_i}$. Now, $\omega_{\alpha_i} = \omega_{\alpha_j} \det(\frac{\partial \vec{x}_\alpha}{\partial \vec{x}_\beta}) = \det(\frac{\partial \vec{x}_\alpha}{\partial \vec{x}_\beta}) > 0$. Thus we are summing positive numbers. □

In other words, M is orientable if and only if $\Omega^m(M)$ is a trivial line bundle. In fact, this can be generalised to : A vector bundle V is orientable if and only if $\Omega^r(V)$ is a trivial line bundle. In the case where V is itself a line bundle, it says that a line bundle is orientable if and only if it is trivial. (Therefore, the Möbius line bundle is not orientable.)

Note that sections of $\Omega^0(M)$ are simply functions $f : M \to \mathbb{R}$. Sections of $\Omega^1(M)$ are 1-form fields. Recall that given a smooth function $f : M \to \mathbb{R}$, there is a natural 1-form - $df : M \to \Omega^1(M)$ defined as $df(X) = X(f)$ where X is a tangent vector, or locally $df = \frac{\partial f}{\partial x^i} dx^i$.

Recall that our aim of doing all of this is to generalise the fundamental theorem of calculus. To do that, we need to take the "curl" $\nabla \times \vec{F}$ of differential forms ω. So in $\mathbb{R}^n$, suppose $\omega = \omega_I dx^I$, the "curl" of it (called the exterior derivative of ω) should naively be "$d \wedge \omega$" $= (\frac{\partial}{\partial x^1} dx^1 + \frac{\partial}{\partial x^2} dx^2 + \ldots) \wedge \omega_I dx^I = \frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I$ (where the summation is over all indices, whether increasing or not). We drop the "$\wedge$" from the above expression and define $d\omega = \frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I = d\omega_I \wedge dx^I$ in $\mathbb{R}^n$.

To make this definition on a mnaifold, we need to make sure that it is independent of coordinates, i.e., if $\omega = \omega_I dx^I$ and in another coordinate system, $\omega = \tilde{\omega_J} d\tilde{x}^J$, then is $d\omega$ well-defined ? That is, is $\frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I = \frac{\partial \tilde{\omega}_I}{\partial \tilde{x}^j} d\tilde{x}^j \wedge d\tilde{x}^I$? We could do this by looking at how $\tilde{\omega}_I$ is related to ω_I but there is a more elegant way : We prove some properties of d and show that any other thing that satisfies these properties coincides with our definition on any open set of $\mathbb{R}^n$ (and therefore is well-defined on a manifold). Indeed,

Proposition 2.2. *If $\omega_1, \omega_2, \omega : U \subset M \to \Omega^K(M)$ are smooth forms where (x, U) is a coordinate chart, then*

(1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
(2) *If ω_1 is a k-form, then* $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$
(3) $d(d\omega) = 0$ *($d^2 = 0$).*

Proof. (1) Trivial (addition rule of partial derivatives).

(2)

$$d(\omega_{1I}\omega_{2J} dx^I \wedge dx^J) = d(\omega_{1I}\omega_{2J}) \wedge dx^I \wedge dx^J = d\omega_{1I} \wedge dx^I \omega_{2J} \wedge dx^J + d\omega_{2J} \wedge \omega_{1I} dx^I \wedge dx^J$$
$$= d\omega_1 \wedge \omega_2 + (-1)^k d\omega_{2J} dx^J \wedge \omega_1 \tag{2.1}$$

(3)

$$d(d\omega_I \wedge dx^I) = d(d\omega_I) \wedge dx^I + (-1)^{k+1} d(dx^I) = 0 \tag{2.2}$$

by induction if we prove it for 1-forms. Indeed, $d(df) = d(\frac{\partial f}{\partial x^i} dx^i) = \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = -\frac{\partial^2 f}{\partial x^j \partial x^i} dx^i \wedge dx^j = -\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j$. Thus $d^2 f = 0$.

□

These properties characterise d on $U \subset M$ (where (x, U) is a coordinate chart).

Proposition 2.3. *Suppose d^a is another map that takes smooth k-forms on U to smooth $k+1$-forms on U for all k and satisfies*

(1) $d^a(\omega_1 + \omega_2) = d^a\omega_1 + d^a\omega_2$
(2) *If ω_1 is a k-form, then* $d^a(\omega_1 \wedge \omega_2) = d^a\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d^a\omega_2$
(3) $d^a(d^a\omega) = 0$ *($(d^a)^2 = 0$).*
(4) $d^a f = df$

then $d^a\omega = d\omega$ on U.

Proof. $d^a(\omega) = d^a(\omega_I dx^I) = d^a\omega_I \wedge dx^I + (-1)^k \omega_I d^a dx^I = d\omega$ by induction if we prove it on functions as well as prove that $d^a(dx^I) = 0$. But for functions it is assumed and $d^a(dx^{i_1} \wedge \ldots) = d^a(d^a x^{i_1} \wedge \ldots) = 0$ by induction. □

Finally, here is a corollary that shows that our d is well-defined.

Corollary 2.4. *There is a unique operator d from the smooth k-forms on M to $k+1$-forms on M for all k satisfying*

(1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

(2) *If* ω_1 *is a k-form, then* $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$
(3) $d(d\omega) = 0$ *(*$d^2 = 0$*).*

and agreeing with the old d on functions.

Proof. For each coordinate chart (x, U) we have a unique d_U defined earlier. Thus define $d\omega(p) = d_U\omega(p)$. ☐

There is yet another way (a third way) to prove that our d is well-defined.

Theorem 2.5. *If* ω *is a smooth k-form on* M*, then there is a unique* $k+1$*-form* $d\omega$ *on* M *such that for every set of vector fields* $X_1, \ldots, X_{k+1}$ *we have*

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots)) + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots) \tag{2.3}$$

where $\hat{X}_i$ *indicates omission. This* $k+1$*-form agrees with the* $d\omega$ *defined previously.*

Proof. Note that in order to define a $k+1$-form, it is enough to define an operator that alternatingly takes $k+1$ vector fields to smooth functions and is linear over smooth functions. It is trivial to see that the above definition satisfies all the properties except perhaps for the linearity of smooth functions. (That requires work.) Replace X_a by fX_a for some fixed a.

$$d\omega(X_1, \ldots, f_a X_a, X_{k+1}) = \sum_{i=1, \neq a}^{k+1} (-1)^{i+1} X_i(f_a \omega(X_1, \ldots, X_a, \ldots, \hat{X}_i, \ldots)) + (-1)^{a+1} f_a X_a(\omega(X_1, \ldots, \hat{X}_a, \ldots)) + likewise \tag{2.4}$$

Noting that $X_i(fg) = X_i f g + f X_i g$, $[fX, Y] = f[X, Y] - (Yf)X$, and $[X, fY] = f[X, Y] + (Xf)Y$ we see that f pulls out.

To check that this coincides with our old d, choose a coordinate sytem (x, U). $d\omega = (d\omega)_J dx^J$ where $(d\omega)_J = d\omega(\frac{\partial}{\partial x^{j_1}}, \ldots)$. Thus

$$(d\omega)_I = \sum_{i=1}^{i+1} (-1)^{i+1} \frac{\partial}{\partial x^{j_i}} (\omega(X_1, \ldots, \hat{X}_i, \ldots)) = \sum_i (-1)^{k+1} \frac{\partial}{\partial x^{j_i}} \omega_{j_1, \ldots, \hat{j}_i, \ldots} \tag{2.5}$$

Thus

$$d\omega = \sum_{i,J} (-1)^{i+1} \frac{\partial}{\partial x^{j_i}} \omega_{j_1, \ldots, \hat{j}_i, \ldots} dx^J = \sum_{i,J} \frac{\partial}{\partial x^{j_i}} \omega_{\tilde{J}} dx^{j_i} dx^{\tilde{J}} = d_U \omega \tag{2.6}$$

☐

In the particular case when ω is a 1-form, the formula says $d\omega = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$. This is useful sometimes. In fact the Frobenius theorem can be stated in the language of forms instead of vector fields, but we won't bother with it.