NOTES FOR 3 NOV (FRIDAY)

1. Recap

(1) Defined closed and exact forms and by means of an example, showed that the topology of a domain can be detected by means of the question "Which closed forms are exact ?"
(2) Stated (and gave an idea of the proof) of Poincaré 's lemma, i.e., on any star-shaped domain (in fact, on any smoothly contractible manifold), every smooth closed form is exact.

2. Integration of top forms over manifolds

If $\omega = f dx^1 \wedge dx^2 \ldots dx^m$, then we may define $\int_{\mathbb{R}^m} \omega = \int f dx^1 dx^2 \ldots$. If you change coordinates to $\tilde{x}$, then $\int \omega = \int f \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^1 \ldots$. The change of variables formula implies that $\int \omega$ is well-defined only if the coordinate changes do not change the sign of the Jacobian, i.e., they are orientation-preserving. But other than that, the integral of a form does not seem to depend on the coordinates chosen.

Definition 2.1. So suppose ω is a smooth top form with compact support (otherwise it may not be integrable) on an oriented manifold M. Assume that $(\vec{x}_\alpha, U_\alpha)$ is a locally finite coordinate cover that is compatible with the given orientation (that is, $\mu_p = [\frac{\partial}{\partial x^1_\alpha}(p), \frac{\partial}{\partial x^2_\alpha}(p), \ldots])$. Let ρ_α be a partition-of-unity subordinate to the open cover. Then define

$$\int_M \omega = \sum_\alpha \int_{\Phi_\alpha(U_\alpha)=\mathbb{R}^m} \rho_\alpha \omega_\alpha dx^1_\alpha dx^2_\alpha \ldots \tag{2.1}$$

Note that $\rho_\alpha \omega_\alpha$ has compact support in $\mathbb{R}^m$ and hence the integral makes sense.

If we choose a different coordinate cover $(\vec{y}_\beta, V_\beta)$ with a different partition-of-unity ψ_β, then we need to prove that we get the same integral.

To do this, firstly, we claim that $\int_{\mathbb{R}^m} \omega_\alpha dx^1_\alpha \ldots = \int \omega_\beta dx^1_\beta \ldots$ whenever ω has compact support in $\mathbb{R}^m$. Indeed, this follows from the change-of-variables formula, orientation-compatibility, and the transformation rule for top forms. Thus $\int_{U_\alpha} \omega$ is well-defined independent of the coordinate chart chosen as long as ω has compact support in U_α.

Secondly, if ω has compact support within one coordinate chart U_α, then $\int_M \omega = \int_{U_\alpha} \omega$. Indeed, if $\int_M \omega = \displaystyle\sum_\beta \int_{V_\beta} \psi_\beta \omega = \sum_\beta \int_{U_\alpha \cap V_\beta} \psi_\beta \omega = \int_{U_\alpha} \omega$ where the last equality holds because, on $\mathbb{R}^m$, $\int_{\mathbb{R}^m} f = \int \sum \psi_\beta f$.

Thirdly, it is easy to see that $\int_M (\omega_1 + \omega_2) = \int_M \omega_1 + \int_M \omega_2$ if both forms have compact support inside one oriented coordinate chart. Indeed, this follows from the addition property of Lebesgue integration for functions in $\mathbb{R}^m$.

Hence, $\int_M \omega = \displaystyle\sum_\alpha \int_M \rho_\alpha \omega = \sum_{\alpha,\beta} \int_M \rho_\alpha \psi_\beta \omega = \sum \beta \int_M \psi_\beta \omega$.

Therefore the integral of top forms with compact support over manifolds is well-defined. Now suppose $f : M \to N$ is a diffeomorphism and suppose ω is a top form on N having compact support, then clearly $f^*\omega$ is a top form on M having compact support. Suppose (y_α, V_α) is an oriented coordinate cover of N and ρ_α is a partition-of-unity. If f is orientation-preserving, then

$(x_\alpha = y_\alpha \circ f, U_\alpha = f^{-1}(V_\alpha)$ is an oriented open cover of M (and $\psi_\alpha = \rho_\alpha \circ f$ is a partition of unity). Thus $\int_M f^*\omega = \sum \int \psi_\alpha f^*\omega = \sum \int f^*(\rho_\alpha \omega)$. By the change of variables formula and the formula for the pullback of a top-form, we see that the latter integral is $\int \omega$. On the other hand, if f is orientation-reversing, we would have gotten $-\int \omega$.

Now we define integration over manifolds with boundary. Firstly, a smooth form ω on a manifold with boundary is a continuous form which is smooth on the interior, and in every boundary chart $\Phi : U \to \mathbb{H}^m$, it extends smoothly to a neighbourhood of the upper half space. (Essentially, a smooth function on a closed set is simply a smooth function on some open set containing the closed set.)

Firstly, an oriented manifold-with-boundary $(M, \partial M, \mu)$ is simply one which can be covered with charts so that the jacobians are positive. (Even for the boundary charts.) So if ω is a smooth top-form on an oriented manifold with boundary $(M, \partial M)$, then we define the integral as follows : Firstly, if ω is a smooth form on $\mathbb{H}^m$ with compact support, then $\int_{\mathbb{H}}^m \omega$ is independent of the coordinates chosen (as long as the coordinates are compatible with the orientation) by the change of variables formula. If ω is a smooth form on M with compact support, then define $\int \omega = \sum \int_{\Phi_\alpha(U_\alpha)} \omega \rho_\alpha$ where ρ_α is a partition-of-unity. As before, this definition is independent of choices.

Before we state Stokes' theorem, let's look at $\int_{[0,1]^m} d\omega$ where $\omega = \omega_1 dx^2 \wedge dx^3 \ldots + \omega_2 dx^1 \wedge dx^3 \wedge \ldots$.

$$\int_{[0,1]^m} d\omega = \sum_i \int_{[0,1]^m} \frac{\partial \omega_i}{\partial x^i}(-1)^{i-1} dx^1 \wedge \ldots dx^m$$

$$= \sum_i \int_{[0,1]^{m-1}} \omega_i(x^1, \ldots, 1, x^{i+1}, \ldots) - \omega_i(x^1, \ldots, 0, \ldots) dx^1 \wedge \ldots \hat{dx^i} dx^{i+1} \ldots \tag{2.2}$$

So we have reduced the integral to an integral over the boundary which is a sum of integrals over each of the boundary faces. However, there seem to be some orientation choice on the boundary. Indeed, suppose we look at the face $x^1 = 1$, then we are integrating $\omega_1(1, x^2, \ldots)$ over it. Whereas for $x^1 = 0$, we pick up a negative sign. This means, $(x^2, \ldots, x^m)$ is an oriented chart for $x^1 = 1$ but it has the opposite orientation for $x^1 = 0$. This means that our orientation on the boundary is by an ordered basis $e_1, \ldots, e_{m-1}$ such that $[N, e_1, \ldots, e_{m-1}] = \mu$ where N is the outward pointing normal.

Using this observation we define the "induced orientation" $\partial\mu$ on ∂M given an orientation μ on M. Suppose we choose a boundary chart $\Phi : U \to \mathbb{H}^m$. Then $[v_1, \ldots, v_{m-1}] \in \partial\mu$ if and only if $[w, v_1, \ldots, v_{m-1}] \in \mu$ where w is any outward pointing vector (pointing towards the lower half-plane).