NOTES FOR 4 OCT (WEDNESDAY)

1. What the heck is a pushforward f_* ? What is df ? What is a pull-back f^*? What is the difference ?

Now since $f_* : T_pM \to T_{f(p)}N$ makes sense for all p, we can use it to define a map $f_* : TM \to TN$ as $(p, v) \to (f(p), f_*v)$. There is no deep difference between these two concepts.

As a special case, if $f : M \to \mathbb{R}$ is a smooth function, then $f_* : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is simply, in coordinates, $f_*(v) = \sum_i v_i \frac{\partial f}{\partial x^i}$. Since f_* acts linearly on tangent vectors and produces real numbers, it is a linear functional on every tangent space. So it makes sense to think of f_* in this case as a section of the cotangent bundle. Therefore, we define $df(v) = f_*v$. In coordinates, $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$. It is a 1-form. Note that given any locally defined function $f : U \subset M \to \mathbb{R}$, $df : U \to T^*M$ is a locally defined 1-form.

So much for the pushforward. Given a smooth map $f : M \to N$ and given a 1-form ω, we can define a 1-form $f^*\omega$ on M. In the earlier classes we defined this locally. Now that we know ω is a linear functional on the tangent space, given a linear map $T : V \to W$, we know that it induces a canonical map $T^* : W^* \to V^*$. We can use this construction here. $f^*\omega(v) = \omega(f_*v)$. So suppose $\omega = \sum_i \omega_i dy^i$, then $(f^*\omega)_p = \sum_j \eta_j(p) dx^j$ where $\eta_j(p) = (f^*\omega)_p(\frac{\partial}{\partial x^j}) = \omega_{f(p)}(f_* \frac{\partial}{\partial x^j}) = \sum_i \omega_i(f(p)) \frac{\partial f^i}{\partial x^j}$.

2. Flow of a vector field - why should I care ?

A vector field $X : M \to TM$ is a section of the tangent bundle. For every smooth vector field, through every point p, there exists a curve $\gamma(t)$ such that its tangent at p is $X(p)$, i.e., $\gamma_*(\frac{\partial}{\partial t})(p) = X(p)$. This is called an integral curve of the vector field through p (or flow through p). For smooth vector fields, the flow is unique. If the vector field has compact support, then the flow is defined for all time through all points. Therefore, such vector fields give rise to a one-parameter family of diffeomorphisms $\phi_t : M \to M$.

So why bother studying vector fields ? They give us the easiest possible way to construct diffeomorphisms. That is the point. Note that you can construct non-trivial examples of compactly supported vector fields easily (using bump-functions). So you can construct non-trivial diffeomorphisms easily. One of our HW problems used this to prove that given any two points p, q on a connected manifold, there is a diffeomorphism so that $\phi(p) = q$. (Try doing this without vector fields. You can't!)

3. Frobenius theorem

Recall that we proved that suppose $X_1, \ldots, X_k$ are smooth vector fields that are linearly independent at p and $[X_i, X_j] = 0$, then locally near p there is a coordinate system such that $X_i = \frac{\partial}{\partial x^i}$, i.e., the X_i are "tangent" to the local submanifold given by $x^{k+1} = \ldots = x^m = 0$.

A natural question is, suppose $[X_i, X_j] \neq 0$, then what can we say ? Nothing much actually! However, if we weaken the condition to $[X_i, X_j](q) = \sum_l C_{ijl}(q) X_l(q)$ (this is read as "the distribution defined by X_i is integrable") for all q near p where C_{ijl} are smooth functions, then the Frobenius theorem states that there are local submanifolds such that X_i are tangent to them, i.e., there exists a coordinate system such that the local embedded submanifold given as a level set $x^{k+1} = a, x^{k+2} = b, x^{k+3} = c\ldots$ is an integral submanifold, i.e., $X_i = \sum_{j=1}^{k} A_{ij}\frac{\partial}{\partial x^j}$.

More generally, given a k-dimensional distribution (rank-k subbundle of TM), i.e., a collection of subspaces $\Delta_p \subset T_pM$ such that around every p, there are k smooth vector fields that are linearly independent everywhere $X_1, \ldots X_k$ spanning Δ_p, an immersed submanifold S is said to be an integral submanifold if $i_* TS_p = \Delta_p$. Frobenius states that if the distribution is integrable, then locally, there are integral submanifolds.

4. Einstein summation convention

I have avoided using the index notation that I used so far in the class. But you could clearly see that not using the notation had two problems :

(1) You had to keep writing the $\sum$ sign over and over.
(2) It was not clear when an expression was coordinate independent, i.e., if $v \in V$ and $w \in W$, $\sum_i v_i w_i = w(v)$ is coordinate independent, but it was not obvious from the choice of notation.

The index notation/Einstein summation makes this obvious. It is practically impossible to write things that are not coordinate independent if you use this notation correctly. As a bonus, you get rid of all the Σs.

So the convention is as follows : Whenever you see a subscript and a superscript having the same index (same alphabet), you sum over that index. For example, suppose V is a vector space and $e_1, e_2, \ldots, e_n$ is an arbitrarily chosen basis. Then any vector v is a linear combination of these vectors. So $v = v^i e_i$ (which should be interpreted as $\sum_{i=1}^{n} v^i e_i$). You should think of things having superscripts as column vectors and things having subscripts as row vectors. So the components of v form a column $\begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ \vdots \end{bmatrix}$. So in $\mathbb{R}^m$, every point is of the form $\vec{x} = (x^1, \ldots, x^m)$. Given a basis e_i for V, there is a dual basis ω^i for V^* (duality changes subscripts to superscripts and vice-versa). So every linear functional w is of the form $w = w_i \omega^i$. So $w(v) = w_i v^i$. Note that whenever you sum over a subscript and a superscript, you get something that is coordinate independent. The w_i are to be thought of as a row vector. Suppose $T : (V, e_i) \to (W, f_\mu)$ is a linear map, then $T(v) = T(v^i e_i) = v^i T(e_i) = v^i T_i^\mu f_\mu$. Matrix multiplication is $(AB)_i^j = A_i^k B_k^j$. Trace is $tr(A) = A_i^i$. Thus $tr(AB) = A_i^k B_k^j = B_k^j A_i^k = tr(BA)$.

In the context of differential geometry, it makes sense to write a vector field as $X = X^i \frac{\partial}{\partial x^i}$. (Note that $\frac{\partial}{\partial x^i} = e_i$ has a subscript.) A one-form is $\omega = \omega_i dx^i$. $\omega(X) = \omega_i X^i$. If you use a different coordinate system y, then $\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$ and $dx^i = \frac{\partial x^i}{\partial y^j} dy^j$. How do the components change ? You can guess (and easily verify) that $X^i = Y^j \frac{\partial x^i}{\partial y^j}$ and $\omega_i = \tilde{\omega}_j \frac{\partial y^j}{\partial x^i}$.

5. Questions you should be able to answer by the end of this course

(1) What is a manifold ? Why should you care ? Give examples.
(2) What is a submanifold ? How to come up with examples of submanifolds ? Is every manifold a submanifold of $\mathbb{R}^N$?
(3) What is the meaning of a smooth map between manifolds ? Can you construct non-trivial maps between manifolds ? What about diffeomorphisms ? What is a regular value ? Why should you care ? How many regular values can an arbitrary map have ?

(4) What is a vector bundle ? What are examples of vector bundles ? Why should you care about the tangent and cotangent bundles ?
(5) What is a vector field and a flow ? Why should you care ? Given a bunch of linearly independent vector fields, can you find a local submanifold that is tangent to these fields ? What is the pushforward of a vector field ?
(6) What is a differential form ? What is a pullback ? Why should you care about forms ? How do you define their integral ? What is d ? What is Stokes' theorem ?
(7) What is a Lie group ? Examples ? subgroup ? Lie algebra ? Why should you care about these things ?