NOTES FOR 4 SEPT (MONDAY)

1. Recap

(1) Gave examples of tangent bundles. (In particular, we stated the Hairy Ball theorem.)
(2) Gave a couple of examples of vector fields. (The constant vector field and the "curling" vector field.)

2. Vector fields, Tangent bundle, Cotangent bundle, etc

Now let us look at (more) examples of vector fields in $\mathbb{R}^n$:

(1) $\vec{X}(x,y) = (x,y)$ is a radial vector field.
(2) $\vec{X}(x,y,z) = (-y,x,0)$ is tangent to the spheres and indeed induces a vector field on the unit sphere. It is *not* nowhere vanishing.

The point of introducing the tangent bundle and vector fields was to model the problem of hydrodynamics on manifolds. So here is a definition :

Definition 2.1. If $X : M \to TM$ is a smooth vector field, then "the flow/integral curve of the vector field through a point $p \in M$" is a smooth curve $c : (-a,a) \to M$ such that $X(c(t)) = c_*(\frac{\partial}{\partial t})$ and $c(0) = p$, i.e., it is a curve passing through p whose tangent vector at every point is the vector field at that point. More concretely, choosing coordinates, an integral curve is locally a function $\vec{c} : (-a,a) \to \mathbb{R}^m$ satisfying the system of ODE : $\frac{dc^i}{dt} = X^i(\vec{c}(t))$, $\vec{c}(0) = \vec{p}$.

Does a flow exist for every vector field at every point $p \in M$? Is it defined on all of $\mathbb{R}$ for every point p ? These questions are non-trivial. Before answering them let us look at examples in $\mathbb{R}^n$. (We will look at general manifolds later.)

(1) The flow of the constant vector field $\vec{X}(x,y) = (1,1)$ is the family of curves satisfying $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 1$, i.e., $x = t + x_0, y = t + y_0$. This flow exists for all time at all points. (By the way, vector fields whose flows at all points exist for all time are called "complete" (as opposed to "stopping" at some point (because of a "blow up" perhaps) and thus being "incomplete").)
(2) The flow of $\vec{X} = (\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}})$ should be a collection of circles (even without solving the ODE) because the vector field is tangent to circles. Indeed, choose polar coordinates to see that $\frac{dr}{dt} = 0$ (thus $r(t) = r_0$) and $\frac{d\theta}{dt} = 1$ (thus $\theta(t) = \theta_0 + t$). This shows that it might be beneficial if we can (at least locally) identify a manifold such that the vector field is tangent to the manifold (so that we can choose nice coordinates to solve the ODE). Does such a manifold even locally exist for every vector field ? We will answer this question later. This is also a complete vector field.
(3) If $\vec{X} = (x^2, 0)$, then $\frac{dx}{dt} = x^2$, $\frac{dy}{dt} = 0$, i.e., $x = \frac{1}{A-t}$, $y = y_0$. So this flow cannot be extended smoothly past $t = A$. It "blows up" in finite time. It is an incomplete vector field. (This raises an interesting question : Suppose the flow remains bounded, then can it be extended for all of time to come ? The answer is yes.)

Suppose a flow exists at every point $p \in M$ for a uniform time interval $(-a, a)$. Suppose also that the image $\phi_t(p)$ of a every point p obtained by following the flow starting at p for time $t \in (-a, a)$ depends smoothly on p. Then what we might get is (hopefully) a family of diffeomorphisms $\phi_t : M \to M$ parametrised by t. Note that so far, we have no method of constructing non-trivial diffeomorphisms of manifolds. (So for example, we could not have answered questions like "Is there a diffeomorphism taking a point p to a point q ?") This is a nice way of coming up with diffeomorphisms - put a fluid on the manifold and see where every point lands after one second. For all of this to make sense, we need to prove a theorem.

Theorem 2.2. *Let X be a smooth vector field on M, and let $p \in M$. Then there is an open set V containing p and an $\epsilon > 0$ such that there is a unique collection of diffeomorphisms $\phi_t : V \to \phi_t(V) \subset M$ for $|t| < \epsilon$ with the following properties :*

1. *$\phi : (-\epsilon, \epsilon) \times V \to M$ defined by $\phi(t, p) = \phi_t(p)$ is smooth.*
2. *If $|s|, |t|, |s + t| < \epsilon$, and $q, \phi_t(q) \in V$, then $\phi_{s+t}(q) = \phi_s \circ \phi_t(q)$.*
3. *If $q \in V$, then X_q is the tangent vector at $t = 0$ of the curve $t \to \phi_t(q)$.*
4. *If X has compact support (in particular, if M is compact), then there are diffeomorphisms $\phi_t : M \to M$ for all $t \in \mathbb{R}$ with the aforementioned properties. In this case, the map $\mathbb{R} \to Diff(M)$ is a group homomorphism called the one parameter group associated to the vector field X.*

The proof of this theorem is non-trivial. Firstly, here is a local result :

Theorem 2.3. *If $\vec{X} : \mathbb{R}^m \to \mathbb{R}^m$ is a smooth vector field, then a flow exists (and is unique) through every point.*

Proof. We need to prove that the following system of ODE with the initial condition $\vec{c}(0) = \vec{p}$ has a unique smooth solution $\vec{c} : (-\epsilon_p, \epsilon_p) \to \mathbb{R}^m$.

$$\frac{d\vec{c}}{dt} = \vec{X}(\vec{c}(t)). \tag{2.1}$$

We can simply quote the existence and uniqueness theorem of ODE, but let us prove that in this case from scratch just for fun.

Let $C = C[[-a, a], \mathbb{R}^m]$ be the space of continuous curves on $[-a, a]$ equipped with the norm $||c_1 - c_2|| = \max_{t \in [-a,a]} |c_1(t) - c_2(t)|$. Define the map $T : C \to C$ given by $T(c)(t) = p + \int_0^t X(c(s))ds$. Note that if $T(c) = c$ then by the fundamental theorem of calculus, c is differentiable and $c' = X(c(t))$ thus solving the ODE. By the chain rule c is twice differentiable (the RHS of the ODE is now differentiable) and so on - c is smooth.

So we need to prove that T has a fixed point. The contraction mapping principle (which applies in this case because C is a complete metric space) shows that we simply need to prove that $||T(c_1) - T(c_2)|| \leq K||c_1 - c_2||$ where $K < 1$. However, this is too much to prove. Instead, take the closed unit ball B around the constant curve $c(t) = p$. It is easy to see that if a is chosen to be small enough, the image of T lies in B. Thus we shall show that $T : B \to B$ has a fixed point. Indeed,

$$\begin{aligned} ||T(c_1) - T(c_2)|| = ||\int_0^t X(c_1) - X(c_2) ds|| &\leq \int_0^a |X(c_1) - X(c_2)| ds \\ &\leq a||X(c_1) - X(c_2)|| \leq aC||c_1 - c_2|| < 1 \end{aligned} \tag{2.2}$$

if a is chosen to be small enough. Note that the second-to-last inequality follows from the mean-value-theorem. □