NOTES FOR 6 NOV (MONDAY)

1. Recap

(1) Defined the integral of a top-form on a manifold using a partition-of-unity and oriented coordinate charts. Proved that it is well-defined, i.e., independent of choices.
(2) Defined the concepts of orientation, tangent bundle, smooth forms, integration, etc for manifolds-with-boundary.
(3) Defined the induced orientation on the boundary of an oriented manifold-with-boundary.

2. Integration of top forms over manifolds

With this convention we state and prove Stokes' theorem.

Theorem 2.1. *If M is an oriented m-dimensional manifold-with-boundary, ∂M is given the induced orientation, ω is an $m-1$-form with compact support, then $\int_M d\omega = \int_{\partial M} \omega$.*

Proof. Let U_α be an oriented cover of M and ρ_α a partition-of-unity subordinate to it.

$$\int_M d\omega = \int_M d(\sum_\alpha \rho_\alpha \omega) = \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) \tag{2.1}$$

Suppose $U_\alpha \simeq \mathbb{R}^m$ and $\eta = \rho_\alpha \omega$ has compact support, then $\int_{\mathbb{R}^m} d\eta = \int_{[-a,a]^m} d\eta = 0$ by the calculation we did above. If $U_\alpha \simeq \mathbb{H}^m$, then the integral is $\int_{[0,a]\times[-a,a]^{m-1}} d\eta = \int_{[-a,a]^{m-1}} \eta(a, x^2, \ldots) - \eta(0, x^2, \ldots) dx^2 \ldots = -\int_{\partial M} \eta(0, x^2, \ldots) = \int_{\partial M} \eta$. Therefore,

$$\int_M d\omega = \sum_{\alpha \in boundary} \int_{\partial M} \rho_\alpha \omega = \int_{\partial M} \omega \tag{2.2}$$

□

Remark 2.2. The above proof is deceptively simple. It appears that we have given a "clean" proof of let's say Green's theorem for a region on the plane whose boundary is a smooth simple closed curve. But that requires a little more work !(Why?)

Remark 2.3. Also, one can make sense of the above statement and the proof for "piecewise smooth" manifolds-with-boundary. (I invite you to define that term for yourself.)

Stokes' theorem is also useful for manifolds without boundary (the usual ones). In that case, $\partial M = \phi$. Therefore $\int_M d\omega = 0$. Suppose ω is an orientation form on a compact manifold M, i.e., ω is nowhere vanishing and the orientation on M is defined as $[v_1, \ldots, v_m] = \mu_p \Leftrightarrow \omega_p(v_1, \ldots, v_m) > 0$. Then $\int_M \omega = \sum \int \rho_\alpha \omega > 0$. Note that every top form is closed. If $\omega = d\eta$, then by Stokes, $\int_M \omega = \int_M d\eta = 0$ but it isn't! Therefore, there are non-exact top forms on M! Therefore M is not contractible ! In other words, compact orientable manifolds are not contractible!

Now we will prove a generalisation of Poincaré 's lemma in a clean and rigorous manner. Before, that here are a couple of definitions :
Suppose M is a smooth manifold (always without boundary unless specified otherwise). Define $i_t : M \to M \times [0,1]$ is $i_t(p) = (p,t)$ and $j_p : [0,1] \to M \times [0,1]$ as $j_p(t) = (p,t)$.

Definition 2.4. Suppose η is a k-form on $M \times [0,1]$, then $K\eta$ is a $(k-1)$-form on M given by :

$$K\eta(X_1, \ldots, X_{k-1}) = \int_0^1 \eta(j_{p*}\frac{\partial}{\partial t}, i_{t*}X_1, i_{t*}X_2, \ldots)dt.$$

where $X_1, \ldots X_{k-1}$ are smooth vector fields on M. This K is called a chain homotopy operator or sometimes, the pushforward and denote as π_* if $\pi : M \times [0,1] \to M$ is the projection.

Theorem 2.5. *For any smooth k-form α on $M \times [0,1]$, we have*

$$i_1^*\alpha - i_0^*\alpha = d(K\alpha) + K(d\alpha)$$

Corollary 2.6. *On a smoothly contractible manifold M, i.e., there is a point $p_0 \in M$ and a smooth map $H : M \times [0,1] \to M$ such that $H(p,1) = p$ and $H(p,0) = p_0$, every closed form ω is exact.*

Proof. Suppose $\alpha = H^*\omega$, then $i_1^*\alpha\omega = \omega$ and $i_0^*\alpha = 0$. Thus, $\omega = d(K\alpha) + K(d\alpha) = d(KH^*\omega) + K(dH^*\omega) = d(KH^*\omega) + KH^*d\omega = d(KH^*\omega)$. □