NOTES FOR 6 OCT (FRIDAY)

1. Recap of Lie groups

(1) A Lie group is a manifold and a group (where the group operations are smooth). A Lie subgroup is an immersed submanifold and a Lie group such that the inclusion map is a homomorphism. Closed Lie subgroups are embedded submanifolds (Cartan's theorem).
(2) Gave several examples of Lie groups. (Abelian and Matrix examples.) In particular, the last time we discussed $Sp(2n, \mathbb{R})$ in more detail. Moreover we looked at $SU(n)$, $SO(n)$, proved that $SU(2) = S^3$, and looked at the isometry group of space $E(n)$. The last example is a semidirect product.
(3) Showed that the connected component of the identity is a closed normal subgroup.
(4) Defined left and right invariance of vector fields.
(5) Showed that every left-invariant vector field is smooth and that every such field is obtained by left translation a vector from T_eG. This proved that Lie groups are parallelizable.

2. Lie algebras

Recall that a Lie algebra is a vector space V with a bilinear operation $[,] : V \times V \to V$ such that $[X, X] = 0\ \forall\ X$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\ \forall\ X, Y, Z$. An example of a Lie algebra is the algebra of smooth vector fields on a manifold with the Lie bracket as the Lie derivative. This is an infinite-dimensional Lie algebra. Typically, these are much harder to handle the finite-dimensional ones. From now onwards, unless specified otherwise, Lie algebras will be finite-dimensional.

An important example of a Lie algebra is T_eG for any Lie group. (In fact, it turns out that this is the only way to produce finite-dimensional Lie algebras. But that is a non-trivial theorem of Lie.) Indeed, given $X_e, Y_e \in T_eG$, extend them to unique left-invariant vector fields $X, Y : G \to TG$. Then we define the Lie bracket as $[X_e, Y_e] = [X, Y](e)$.

With this definition, it is easy to see that the Lie algebra of $\mathbb{R}^n$ and $S^1 \times S^1 \times \ldots$ is Abelian, i.e., $[X_e, Y_e] = 0\ \forall\ X_e, Y_e$.

Let us calculate the Lie algebras of some other Lie groups :

(1) $\mathfrak{gl}(n, \mathbb{R})$: Suppose A, B are two elements in T_eGL, i.e., they are two $n \times n$ matrices. Assume that the coordinates on GL are x^{ij}. Then $A = A^{ij}\frac{\partial}{\partial x^{ij}}$ and $B = B^{ij}\frac{\partial}{\partial x^{ij}}$. Let $\tilde{A}(a) = L_{a*}A$ and $\tilde{B}(a) = L_{a*}B$ be the corresponding left-invariant extensions. Now $L_a x = ax$ which is a linear map. Therefore, $L_{a*}A = (aA)^{ij}e_{ij}$ and likewise, $\tilde{B}(a) = (aB)^{ij}e_{ij}$. Thus $[\tilde{A}, \tilde{B}](e) = A^{ij}B^{kl}[e_{ij}, e_{kl}] = ([A, B])^{ij}e_{ij}$ where the last expression is the matrix commutator. (Why is this last equality true ?)
(2) Since $O(n)$ is a Lie subgroup of GL, we may identify $\mathfrak{o}(n)$ with a subspace of $\mathfrak{gl}$. Suppose $A, B \in \mathfrak{o}(n) \subset \mathfrak{GL}$. Then $i_*\tilde{A}$ is a left-invariant vector field on $O(n)$ because $L_a \circ i = i \circ L_a$. Thus $\mathfrak{o}(n)$ is a Lie subalgebra of $\mathfrak{gl}$.

More generally, if $H \subset G$ is a Lie group, then $\mathfrak{h} = T_eH \subset \mathfrak{g} = T_eG$ is a subalgebra. Indeed, it is a subspace. All we need to prove is that $i_* : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism. Suppose $a, b \in \mathfrak{h}$. Then $[a, b] = [X, Y]$ where X, Y are the unique left-invariant extensions of a and b on H. There are also unique left-invariant extensions $\tilde{X}, \tilde{Y}$ on G. Since the left translation maps of H, G, and the inclusion

i commute, we see that $i_* X = \tilde{X}$. That $i_*[X, Y] = [\tilde{X}, \tilde{Y}]$ follows from the following basic lemma for embedded Lie groups (actually the following lemma can be generalised considerably but we will first do this just to get practice with coordinates).

Lemma 2.1. *Suppose $i : S \subset M$ is an embedding, and X, Y are smooth vector fields on S such that there are smooth vector fields $\tilde{X}, \tilde{Y}$ on a neighbourhood of S in M satisfying $i_* X = \tilde{X}$, $i_* Y = \tilde{Y}$ on S. Then $[X, Y](p) = [\tilde{X}, \tilde{Y}](p)$*

Proof. We know (by now) that the IFT implies that there exists a coordinate system (y, U) near p on M such that $S \cap U = y^{s+1} = \ldots = y^m = 0$ and $(x^1 = y^1, \ldots, x^s = y^s)$ form coordinates on S such that $i(x^1, \ldots, x^s) = (x^1, \ldots, x^s, 0, \ldots)$. Let $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$, and $\tilde{X} = \tilde{X}^i \frac{\partial}{\partial y^i}$, $\tilde{Y} = \tilde{Y}^i \frac{\partial}{\partial y^i}$. Then $i_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$ for $1 \leq i \leq k$. Therefore,

$$[\tilde{X}, \tilde{Y}] = [\tilde{X}^i \frac{\partial}{\partial y^i}, \tilde{Y}^j \frac{\partial}{\partial y^j}] = \left[\tilde{X}^i \frac{\partial \tilde{Y}^j}{\partial y^i} - \tilde{Y}^i \frac{\partial \tilde{X}^j}{\partial y^i} \right] \frac{\partial}{\partial y^j} \tag{2.1}$$

At $p \in S$, in the summation over i, it ranges only from 1 to k. Moreover, $\tilde{Y}^i$, $\tilde{X}^i$ are zero when $i > k$. Thus we see that $[\tilde{X}, \tilde{Y}](p) = [X, Y](p)$. □