NOTES FOR 6 SEPT (WEDNESDAY)

1. Recap

(1) Gave more examples of vector fields and defined flows.
(2) Gave examples of flows. Defined complete vector fields.
(3) Stated the theorem on existence of flows on manifolds. Proved the local version in $\mathbb{R}^n$ using the contraction mapping principle.

2. Vector fields, Tangent bundle, Cotangent bundle, etc

Proof of a part of the flow existence and uniqueness theorem on manifolds : Even if we grant the local existence and uniqueness theorem, while it is obvious that part 4) of the theorem is true, it is not clear that $\phi(t, \vec{p})$ is smooth. This is a non-trivial result (the proof is in Lang's book on differential geometry). If we grant that, then parts 1) and 4) are done. As for part 2) assume without loss of generality that V is a coordinate ball $B_a(p)$, note that since $\phi : (-\epsilon, \epsilon) \times B_{\frac{a}{2}}(p) \to M$ satisfies $\phi(0, x) = x$ and is smooth, this means that by shrinking ϵ if necessary, we may assume that $\phi : (-\epsilon, \epsilon) \times B_{\frac{a}{2}}(p) \to B_a(p)$. So if $|s| < \epsilon$, and $p \in B_{a/2}(p)$, then $\gamma(t) = \phi(t, \phi(s, p)) = \phi_t \circ \phi_s(p)$ makes sense on $|t| < \epsilon$. Now γ satisfies $\gamma(0)\phi_s(p)$ and the ODE

$$\gamma^{'}(t) = X(\gamma(t)). \tag{2.1}$$

Moreover, the curve $\beta(t) = \phi(s + t, p)$ defined on $|s + t| < \epsilon$ also satisfies, the same ODE with the same initial conditions. By uniqueness, we are done. Also, since $\phi_{-s} = \phi_s^{-1}$, this means that ϕ_s are diffeomorphisms.

As for the last part, note that away from a compact set (the support of X), the flow of the vector field is trivial (i.e. the points do not move). Cover the support of X by finitely many open sets $V_1, \ldots, V_k$ from the previous parts of this theorem. Choose the minimum of the ϵs necessary. Now define $\phi : (-\epsilon, \epsilon) \times M \to M$ as $\phi(t, p) = \phi_i(t, p)$ where if $\phi_i(t, p)$ is the diffeomorphism from V_i if $p \in V_i$ and $\phi_i(t, p) = p$ if p is outside the support of X. I claim that $\phi(t, p)$ is well-defined. Indeed, if $p \in V_i \cap V_j$, then $\phi_i(t, p) = \phi_j(t, p)$ by uniqueness of integral curves. So the ϕ are smooth diffeomorphisms. Since $\phi_{\frac{\epsilon}{2}}$ can be composed with itself arbitrary number of times (because it is a diffeomorphism of M), we can easily extend ϕ to all of $\mathbb{R}$. (Why ?) □

Recall that $\frac{df \circ c}{dt} = [c](f)$ where $[c]$ is thought of as a derivation. Note that this also means $(Xf)(p) = \frac{df \circ c}{dt} = \lim_{h \to 0} \frac{f(\phi_h(p)) - f(p)}{h}$. Now we use the flow existence theorem to prove the following theorem which answers an earlier question of ours.