NOTES FOR 7 AUG (MONDAY)

1. Recap

(1) Gave a biased history of geometry.
(2) Stated an optimisation problem on a sphere. Decided to define sphere-like objects (manifolds) that are locally Euclidean, have smooth functions on them, and have quantities that behave well under coordinate changes.
(3) Defined a topological manifold. Proved the notion of dimension is well-defined. Gave a naive definition of a C^k manifold through C^kness of transition functions. Decided that we need to cut our ability to add more charts. To that end we defined a C^k maximal atlas.

2. Definition of a manifold and examples

Lemma 2.1. *Every C^k atlas $\mathcal{A}$ is contained in a unique C^k maximal atlas $\mathcal{A}'$*

Proof. Consider the set of all C^k atlases containing $\mathcal{A}$ (thanks to Chaitanya for pointing this typo out) S equipped with the above partial order of containment. (This is an honest set in ZFC.) Given any chain $\mathcal{A}_i \subset \mathcal{A}_j \subset \ldots$, let $\mathcal{B} = \cup_i \mathcal{A}_i$. This is certainly an atlas because, given any two charts (U, Φ_U) and (V, Φ_V), there is an atlas A_k containing these two and hence the transition map is C^k. Thus by Zorn's lemma there exists a C^k maximal atlas. If there are two of them, then their union gives a larger atlas. This is a contradiction to maximality. Hence uniqueness holds. □

We now give the correct definition of a C^k manifold.

Definition 2.2. A C^k manifold $(M, \mathcal{A})$ is a topological manifold M equipped with a C^k maximal atlas $\mathcal{A}$. In this case $\mathcal{A}$ is said to provide M with a C^k differentiable structure. (We will abuse notation and forget the $\mathcal{A}$ entirely from now on.)

Remark 2.3. It is a very non-trivial result that for $k \geq 1$, every C^k structure is compatible with a C^∞ structure, i.e., given a C^k maximal atlas $\mathcal{A}$ there exists a C^∞ maximal atlas $\mathcal{A}'$ that is C^k compatible with $\mathcal{A}$. It is almost unique (the correct statement is that it is unique upto diffeomorphism). So the study of C^k manifolds for $k \geq 1$ is reduced to that of C^∞ manifolds. But the shocking thing is that for $k = 0$ this is FALSE. In other words, there are examples of topological manifolds not admitting a smooth structure, and yet others admitting more than one smooth structure ! (In fact $\mathbb{R}^4$ has more than one smooth structure, i.e., more than one way to do calculus!) These exotic structures occur in dimension ≥ 4. It is quite non-trivial to prove that this phenomenon does not occur in lower dimensions. The first exotic structure was found on S^7 by John Milnor (in a 7 page paper in the Annals!). He was awarded the Fields medal for that discovery.

From now onwards, we will only study smooth manifolds. Also, in order to prove something is a manifold, it is enough to provide one atlas. The unique maximal atlas containing it defines the smooth structure. We will now define what it means for two smooth manifolds to be "essentially the same" :

Definition 2.4. Two C^∞ manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ are said to be diffeomorphic if there is a 1-1 onto function $f : M \to N$ such that $(V, \Phi_V) \in \mathcal{B} \iff (f^{-1}(V), \Phi_V \circ f) \in \mathcal{A}$

It is clear that f is a diffeomorphism if and only if f^{-1} is so. It is also easy to see that f is a homeomorphism (why?). Hence the dimensions of M and N are the same. Moreover, suppose $(U, \Phi_U) \in \mathcal{A}$ and $(V, \Phi_V) \in \mathcal{B}$. Then $g_{VU} = \Phi_V \circ f \circ \Phi_U^{-1} : \Phi_U(U) \subset \mathbb{R}^n \to \mathbb{R}^n$ is smooth (and its inverse is smooth) because $(f^{-1}V, \Phi_V \circ f) \in \mathcal{A}$ and hence there g_{VU} is actually a transition function for M and hence is smooth (with its inverse being smooth as well). Conversely, any homeomorphism such that when written in coordinates it is smooth and its inverse is smooth is a diffeomorphism.

Before we proceed to examples of smooth manifolds, here is a small observation : In the definition of a manifold, we may (without loss of generality) require that Φ_α be a homeomorphism of U_α to *all of* $\mathbb{R}^n$ as opposed to an open subset. Indeed, if it is so to an open subset, then surely we can find a cover by smaller open sets $\tilde{U}_\gamma$ such that these are homeomorphic to open cubes in $\mathbb{R}^n$. Now any open cube is homeomorphic by means of a linear map (so it is differentiable and its inverse is differentiable) to $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \ldots$ which is in turn homeomorphic to $\mathbb{R}^n$ via $f(x_1, \ldots, x_n) = (\tan(x_1), \tan(x_2) \ldots)$ (this is a smooth map whose inverse is also smooth).

Here are some examples of smooth manifolds:

(1) Of course $\mathbb{R}^n$ and open subsets of the same are obvious examples that we discussed earlier.
(2) The sphere S^n. Indeed, $x_1^2 + \ldots + x_{n+1}^2 = 1$ is the sphere. Cover it with open sets where $U_i = \{x_i > 0\}$, $V_i = \{x_i < 0\}$. Indeed, every point $(x_1, \ldots, x_{n+1})$ has at least one coordinate x_i non-zero and is hence in either in U_i or V_i. On U_i, choose the other coordinates $x_i = \sqrt{1 - x_1^2 - \ldots - x_{i-1}^2 - x_{i+1}^2 \ldots}$ and so on. Check that indeed on the intersections, the transition functions are diffeomorphisms.
(3) Given two manifolds M, N, $M \times N$ is a manifold. (Why ?) So $S^1 \times S^1 \ldots$ is a manifold. Another example is the cylinder $S^1 \times \mathbb{R}$.
(4) Quotients of manifolds need not always be manifolds. But here are two examples :
  (a) Torus : $T^2 = \frac{\mathbb{R}^2}{\vec{x} \equiv \vec{x} + \sum n_i e_i}$. Indeed, consider the open cover given by $U_1 = \pi((0,1) \times (0,1))$, $U_2 = \pi((-1/2, 1/2) \times (0,1))$, $U_3 = \pi((-1/2, 1/2) \times (-1/2, 1/2))$, $U_4 = \pi((0,1) \times (-1/2, 1/2))$. On U_1, $\Phi_1(\pi(x,y)) = (x,y)$, on U_2, $\Phi_2(\pi(x,y)) = (x + \frac{1}{2}, y)$, etc. It is clear that $\phi_{21}(a,b) = (a + \frac{1}{2}, b)$ is smooth. Likewise, the other transition functions are smooth. The same argument may be generalised to the n-torus. (In fact, the n-torus is diffeomorphic to $S^1 \times S^1 \ldots$. (Why ?)It is clearly homeomorphic to the same. If one wants, one can induce a differential structure using the one on $S^1 \times S^1 \ldots$. It gives the same structure as the one above.)
  (b) Real projective space $\mathbb{RP}^n$: Motivated by perspective art of the 14th century, one can study the set of all lines (light rays) passing through the origin (the eye) in $\mathbb{R}^{n+1}$. Each such line intersects the sphere S^n in exactly two (antipodal) points. Since they correspond to the same line, they ought to be identified to produce $\mathbb{RP}^n = \frac{S^n}{\vec{x} \equiv -\vec{x}}$ with a quotient map π from S^n. This is a smooth manifold. Indeed, consider the open cover $U_i = \pi(\{x_i > 0\} \cap S^n$ with $\Phi_i(\pi(\vec{x})) = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots)$. It is easy to see that this is a C^∞ atlas.
  (c) Complex projective space $\mathbb{CP}^n$: Just as real lines through the origin in $\mathbb{R}^{n+1}$ are a natural object to study, for the purposes of algebraic geometry, complex lines through the origin in $\mathbb{C}^{n+1}$ are useful. Thus we define $\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - \vec{0}}{\vec{X} \equiv \lambda \vec{X} \text{ where } \lambda \in \mathbb{C}^*}$. The equivalence class of

$X_0, X_1, \ldots$ is denoted as $[X_0 : X_1 \ldots : X_n]$. Consider the open sets $U_i = \pi(\{X_i \neq 0\})$. Define $\Phi_i : U_i \to \mathbb{C}^n = \mathbb{R}^{2n}$ as $\Phi_i([X_0 : X_1 \ldots]) = (\frac{X_0}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots)$. The transition functions are easily seen to be smooth. (In fact, if you ever learn several complex variables, you will see that they are biholomorphisms. This is a complex manifold.) Is $\mathbb{CP}^n$ compact ?

(5) 1-manifolds : S^1 is an example of a 1-dimensional manifold. The open interval (a, b) is another example. It turns out that all 1-manifolds "look like" (i.e. are diffeomorphic to) these. This is a non-trivial theorem proven in Milnor's book.

(6) $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$: These are open subsets of $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$ and are hence manifolds.