NOTES FOR 8 NOV (WEDNESDAY)

1. Recap

(1) Proved Stokes' theorem. Saw that deducing the usual Green theorem from our Stokes theorem requires some effort.
(2) Compact orientable manifolds are not contractible.
(3) Defined fibre integration. Saw that a theorem implies Poincaré 's lemma as corollary.

2. Closed and exact forms

Theorem 2.1. *For any smooth k-form α on $M \times [0,1]$, we have*

$$i_1^*\alpha - i_0^*\alpha = d(K\alpha) + K(d\alpha)$$

Proof. Choose a coordinate chart (x, U) near p. Then, $\alpha = \alpha_I(x,t)dt \wedge dx^I + \tilde{\alpha}_J(x,t)dx^J$. Now $K\alpha = (\int_0^1 \alpha_I(x,t)dt)dx^I$. Thus

$$\begin{aligned} d(K\alpha) + K(d\alpha) &= d_x(\int_0^1 \alpha_I(x,t)dt)dx^I + (\int_0^1 ((d_x + d_t)\alpha)_K(x,t)dt)dx^K \\ &= (\int_0^1 (d_x\alpha_I(x,t) \wedge dx^I)dt) + (-\int_0^1 (d_x\alpha_I) \wedge dx^I dt) + \int_0^1 \frac{\partial \tilde{\alpha}_J}{\partial t} dt dx^J \\ &= \tilde{\alpha}_J(p,1) - \tilde{\alpha_J}(p,0) = i_1^*\alpha - i_0^*\alpha \end{aligned} \tag{2.1}$$

□

3. De Rham cohomology

Since we saw that it is important to know which closed forms are exact in order to detect non-trivial topology of manifolds, we define the following vector spaces :

Definition 3.1. Suppose $C^k(M)$ is the vector space of all smooth closed k-forms on M and $E^k(M)$ the space of all exact k-forms, then the quotient space $H^k(M) = \frac{C^k}{E^k}$ is called the kth De Rham cohomology group of M.

It turns out that for compact manifolds, the De Rham cohomology is finite dimensional. (But proving this is quite non-trivial.) Let us compute a few of these groups.

$H^0(M)$: Z^0 is empty and $C^0(M)$ is smooth f satisfying $df = 0$, i.e., f is a constant on each component. Therefore $H^0(M) = \mathbb{R}^c$ where c is the number of connected components of M.

If M is contractible and connected, $H^0(M) = \mathbb{R}$ and all other H^k are 0 (by Poincaré). If M is compact and oriented, then $H^m(M)$ is at least one dimensional. (It turns out to be exactly one-dimensional as we shall see later on.)

Given a map $f : M \to N$, we have the pullback $f^* : \Omega^k(N) \to \Omega^k(M)$. Since the pullback commutes with d, it descends to a map $f^* : H^k(N) \to H^k(M)$. Moreover, since $(f \circ g)^* = g^* \circ f^*$, if f is a diffeomorphism, then $H^k(N) \simeq H^k(M)$. So the De Rham cohomology is a diffeomorphism invariant. (It turns out that it is actually a homeomorphism, or even better, a homotopy invariant.)

Before we go ahead, we define a different cohomology group (De Rham with compact support) : $H^k_c(M) = C^k_c(M)/E^k_c(M)$ where the forms have compact support. If M is compact, then this coincides with the usual De Rham groups. Note that $E^k_c M$ does not consist of all exact forms with compact support. For example, if on $\mathbb{R}^m$, $\omega = f dx^1 \wedge \ldots dx^m$, where f has compact support, then it is exact (Poincaré). But if $\omega = d\eta$ where η also has compact support, then $\int \omega = \int d\eta = 0$ but that is not the case if $f \geq 0$ (and > 0 on an open set). So $H^k_c(\mathbb{R}^m) \neq 0$. In fact,

Theorem 3.2. *If M is a connected orientable m-manifold, then the map $T[\omega] = \displaystyle\int_M \omega$ gives an isomorphism $H^m_c(M) \simeq \mathbb{R}$.*

Proof. We know that T is surjective. Indeed, if ω is an orientation form, and f is a bump function with compact support, then $\int_M f\omega > 0$. Thus $f\omega$ is not exact.

Suppose $\displaystyle\int_M \omega = 0$. We need to prove that $\omega = d\eta$ where η has compact support.

We shall prove the theorem in three steps (induction - True for $\mathbb{R}$, True for $\mathbb{R}^m$ assuming it is true for $m-1$ folds, true for general m folds assuming it is true for $\mathbb{R}^m$.)

(1) True on $\mathbb{R}$ and S^1 : If ω is a 1-form with compact support on $\mathbb{R}$ such that $\int_{\mathbb{R}} \omega = 0$, then define a function $f(x) = \displaystyle\int_{-\infty}^{x} \omega$. Note that $df = \omega$ by the FTC. f has compact support because $f(-a) = 0$ for a large positive a and since $\int_{\mathbb{R}} \omega = \int_{-a}^{b} \omega$ for all $b \geq a$, we see that $f(x) = 0$ for $x > a$. Hence the theorem holds for $\mathbb{R}$. Likewise, if $\omega = g(\theta)d\theta$ where $g(0) = g(2\pi)$, then $f(\theta) = \int_0^\theta g(\theta)d\theta$ satisfies $df = \omega$ and $f(0) = 0 = f(2\pi)$. Thus the theorem holds for the circle too.

TO BE CONT'D.... □