NOTES FOR 8 SEPT (FRIDAY)

1. Recap

(1) Proved that flows exist, are unique, and for compactly supported vector fields, they provide a one-parameter group of diffeomorphisms.

2. Vector fields, Tangent bundle, Cotangent bundle, etc

Theorem 2.1. *Let X be a smooth vector field on M with $X(p) \neq 0$. Then there exists a coordinate system (x, U) around p so that $X = \frac{\partial}{\partial x^1}$ on U.*

Proof. The idea is to simply flow along X and call the integral curves the x^1-coordinate "lines". Indeed, first choose a coordinate system (y, V) around p such that $X(0) = \frac{\partial}{\partial y^1}(0)$, $X, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}, \ldots$ form a basis for the tangents spaces on V, and p corresponds to $y = 0$.

Define $h(x^1, \ldots, x^m) = \phi_{x^1}(0, x^2, \ldots, x^m)$. We claim that this is a local diffeomorphism around $(0, \ldots, 0)$ such that $h_*(\frac{\partial}{\partial x^1} = X$. Therefore x^i provide a new coordinate system on a neighbourhood U of p and do the job. Indeed,

$$h_*(\frac{\partial}{\partial x^1})(f) = \frac{\partial f(\phi_{x^1}(0, x^2, \ldots))}{\partial x^1} = Xf(h(x))$$

$$h_*(\frac{\partial}{\partial x^i})(f) \text{ at } x = 0 = \frac{\partial f}{\partial x^i}(x = 0). \tag{2.1}$$

Thus h is an immersion at p and by the inverse function theorem, it is a local diffeomorphism. □

From now onwards, we denote Xf as $L_X f$ and call it "The Lie derivative of f along X". The reason is that we can define the Lie derivative of other beasts like vector fields and one-forms. Indeed, define

$$L_X Y(p) = \lim_{h \to 0} \frac{Y(p) - ((\phi_h)_* Y)(p))}{h}$$

Likewise, if ω is a one-form, then define

$$L_X \omega(p) = \lim_{h \to 0} \frac{(\phi_h^* \omega)(p) - \omega(p)}{h}$$

Note that $(\phi_h)_* Y)(p) = (\phi_h)_*(Y_{\phi_{-h}(p)})$. The Lie derivative satisfies the following easy linearity properties.

(1) $L_X(\alpha_1 Y_1 + \alpha_2 Y_2) = \alpha_1 L_X Y_1 + \alpha_2 L_X Y_2$
(2) $L_X(\alpha_1 \omega_1 + \alpha_2 \omega_2) = \alpha_1 L_X \omega_1 + \alpha_2 L_X \omega_2$.

In addition, if $L_X Y, L_X \omega$ exist, then so do $L_X(fY), L_X(f\omega)$ where f is smooth, and

(1) $L_X(fY) = XfY + fL_X Y$
(2) $L_X(f\omega) = Xf\omega + fL_X \omega$

Proof. We will prove the first one. The second is similar.

$$L_X(fY)(p) = \lim_{h\to 0} \frac{f(p)Y(p) - (\phi_h)_*(f(\phi_{-h}(p))Y_{\phi_{-h}(p)})}{h}$$
$$= \lim_{h\to 0} \frac{f(p)Y(p) - f(\phi_{-h}(p))(\phi_h)_*(Y_{\phi_{-h}(p)})}{h}$$
$$= Xf(p)Y(p) + f(p)L_XY(p), \tag{2.2}$$

where the last equality follows by adding and subtracting $\frac{f(p)(\phi_h)_*(Y_{\phi_{-h}(p)})}{h}$, and by the easy fact that $\lim(\phi_h)_*Y_{\phi_{-h}(p)} = Y_p$. □

Now we may compute the Lie derivative in local coordinates. Indeed, suppose $X = X^i\frac{\partial}{\partial x^i}$, $\omega = \omega_i dx^i$, and likewise for Y, then

$$L_XY = Y^iL_X\frac{\partial}{\partial x^i} + X(Y^i)\frac{\partial}{\partial x^i}$$
$$L_X\omega = \omega_iL_Xdx^i + X(\omega_i)dx^i. \tag{2.3}$$

So we just have to evaluate L_X on the basis vector fields and 1-forms. Denote $e_i = \frac{\partial}{\partial x^i}$. Now $(\phi_h)_*(e_i)(p)(x^j) = \frac{\partial\phi_h^j}{\partial x^i}(\phi_{-h}(p))$ and $(\phi_h)^*dx^i = \frac{\partial\phi_h^i}{\partial x^j}dx^j = e_j(\phi_h^i)dx^j$ Thus

$$L_Xdx^i = \lim_{h\to 0}\frac{e_j(\phi_h)^idx^j - dx^i}{h} \tag{2.4}$$

By smoothness of $A(h,p) = x^i(\phi_h(p))$, we can interchange derivatives and limits to get

$$L_Xdx^i = dx^je_j\lim_{h\to 0}\frac{(\phi_h)^i - x^i}{h} = dx^je_j(X^i). \tag{2.5}$$

Likewise,

$$L_Xe_i(x^j) = \lim_{h\to 0}\frac{e_i(x^j) - e_i(\phi_h^j)(\phi_{-h})}{h} = \lim_{h\to 0}\frac{e_i(x^j)(\phi_{-h}) - e_i(\phi_h^j)(\phi_{-h})}{h} = e_i\lim_{h\to 0}\frac{x^j - \phi^j}{h} = -e_iX^j \tag{2.6}$$

Thus $L_XY = (e_i(Y^j)X^i - e_i(X^j)Y^i)e_j$. Note that the following is easily proven to be true :

Lemma 2.2. $L_XY(f) = X(Y(f)) - Y(X(f)) = [X,Y](f)$