NOTES FOR 9 AUG (WEDNESDAY)

1. Recap

(1) Gave the correct definition of a C^k manifold and remarked that it is enough to study topological manifolds and C^∞ manifolds. We will study only smooth manifolds in this course.
(2) Defined the notion of a diffeomorphism and characterised it in terms of coordinates.
(3) Gave several examples of smooth manifolds.

2. Definition of a manifold and examples

Definition of a manifold-with-boundary : We want to consider $[0, 1)$ as a manifold, albeit with a boundary 0. This is useful to formulate things like Stokes' theorem and also to solve optimisation problems with boundary conditions. Firstly, we define the closed upper half-space $\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ The definition is as follows :

Definition 2.1. A smooth manifold-with-boundary M is a Hausdorff, second-countable space covered by a maximal collection of open sets U_α such that there exist homeomorphisms Φ_α that either map U_α homeomorphically to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$, and the transition functions are smooth. (A smooth function on a set $E \in \mathbb{R}^n$ is defined to be a smooth function on some open neighbourhood of E.)

Remark 2.2. There is always an atlas (not maximal) such that the open sets U_α are homeomorphic either to all of $\mathbb{R}^n$ or all of $\mathbb{H}^n$ just as in the case of manifolds (without boundary).

Definition 2.3. Suppose M is a smooth manifold-with-boundary. The set of all points $p \in M$ such that there is a neighbourhood $p \in U$ that is homeomorphic to $\mathbb{H}^n$ but have no neighbourhoods homeomorphic to $\mathbb{R}^n$ is called the boundary $\partial M \subset M$ of M. Equivalently (why?) there $p \in \partial M \iff p \in U$ where $\exists$ a homeomorphism (compatible with the smooth structure) $\Phi : U \to \mathbb{H}^n$ such that $\Phi(p) = \vec{0}$.

(Exercise : Prove that ∂M is a smooth $(n-1)$-fold (possibly disconnected) without boundary.)

Remark 2.4. A similar definition can be made for topological manifolds too.

Here are some examples :

(1) 1-manifolds-with-boundary : $[a, b)$, $[a, b]$, and $(a, b]$ are 1-manifolds with boundary. In fact, (look at Milnor's book) all 1-manifolds-with-boundary are diffeomorphic to these.
(2) Any open subset of $\mathbb{H}^n$ intersecting $x_n = 0$ non-trivially.
(3) The upper hemisphere has boundary the $n-1$ sphere. (Why ?)

3. Maps between manifolds, Submanifolds, IFT, etc

So far we defined what a diffeomorphism between two manifolds is : A homeomorphism that takes the maximal atlas of one to that of the other. We saw an alternate characterisation of it in terms of coordinates. We expand upon that further :

Definition 3.1. If M, N are smooth manifolds, then a function $f : M \to N$ is called smooth (or differentiable) if for every pair of coordinate charts $(U \subset M, \Phi_U), (\tilde{U} \subset N, \tilde{\Phi}_{\tilde{U}})$ $\tilde{\Phi}_{\tilde{U}} \circ f \circ \Phi_U^{-1} : \Phi_U(U) \subset \mathbb{R}^m \to \tilde{\Phi}_{\tilde{U}} \subset \mathbb{R}^n$ is a smooth map.

Remark 3.2. Note that one can replace smooth with C^1 or any other such condition to the corresponding definition for such maps. Also note that our alternate condition of a diffeomorphism is simply that f and f^{-1} are smooth.

Remark 3.3. The above definition makes sense even for manifolds-with-boundary. Indeed, define a smooth map on $\mathbb{H}^n$ as simply a smooth map on an open neighbourhood of it.

There are obvious examples of smooth maps like the projection $\pi_1 : M \times N \to M$ and the quotient map $\pi : S^n \to \mathbb{RP}^n$ (why is the latter smooth?). But given a general manifold M, is there even *one* non-constant example of a smooth map $f : M \to \mathbb{R}$? To answer this question, we need a general tool - Bump functions.

The function $h(x) = e^{-1/x^2}$ when $x > 0$ and 0 when $x \leq 0$ is smooth everywhere (including the origin). By shifting and multiplying we get $j(x) = h(x+1)h(1-x)$. In fact, we can repeat this construction $k(x)$ for any interval $(0, \delta)$. Upon integrating and normalising we get $l(x) = \dfrac{\int_0^x k}{\int_0^\delta k}$ which is 1 for $x \geq \delta$ and 0 for $x \leq 0$. The function $t(x) = l(x)l(3\delta - x)$ is an example of a "bump function" which is 1 on $[\delta, 2\delta]$ and 0 on $(-\infty, 0] \cup [3\delta, \infty)$. By shifting and multiplying function $t(x_i)$, we can get bump functions $t(\vec{x})$ with any rectangle-shaped support. In fact, the same thing holds for manifolds :

Lemma 3.4. *Let $C \subset U \subset M$ with C compact and U open. Then there is a smooth function $f : M \to [0,1]$ such that $f = 1$ on C and support $f \subset U$.*

Proof. Each point p of C is in some coordinate open square $B_{p,\epsilon}$ of size ϵ such that $B_{p,2\epsilon} \subset S$. Only finitely many of the $B_{p,\epsilon}$ are required to cover C (by compactness). On each of these $B_{p,\epsilon}$, we can find a bump function $t_p(\vec{x})$ whose support is in $B_{p,2\epsilon}$ and which is 1 on $B_{p,\epsilon}$. These t_p can be extended to the entire manifold by setting them to be 0 outside $B_{p,2\epsilon}$. Let $t = t_{p_1} + t_{p_2} \ldots$. This is a finite sum of non-negative functions that is $\geq \delta > 0$ on C and is 0 outside U. Let $f = l \circ t$. This gives the desired function. ◻

Now that we know what smooth maps are, the next order of business is actually differentiating them. (Recall that we started this story with an example "Given a smooth function P on a sphere, find its maximum." This example required us to differentiate the function P (the final answer did not depend on the coordinates chosen).)

Definition 3.5. Suppose $f : N \to \mathbb{R}$ is a smooth map. If (U, Φ_U) is a coordinate chart on N (where $\Phi_U(p) = (x^1(p), x^2(p), \ldots)$; we will use superscripts from now on to indicate coordinates because of a reason that we will see later on), then define the partial derivatives in these coordinates, denoted as $\frac{\partial f}{\partial x^i}$ as $\frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \Phi_U^{-1})}{\partial x^i}(\Phi_U(p))$.

Remark 3.6. Here is a trivial observation : Suppose we define a curve $\vec{c} : (-\epsilon, \epsilon) \to N$ as $\vec{c}(t) = \Phi_U^{-1}(\vec{x}(p) + (0, \ldots, t, 0 \ldots))$ then $\lim_{h \to 0} \frac{f(\vec{c}(h)) - f(p)}{h} = (\Phi_U \circ f \circ \vec{c})'(0) = \frac{\partial f}{\partial x^i}(p)$.

Another trivial observation is that $\frac{\partial x^i}{\partial x^j} = \delta^i_j = 1$ when $i = j$ and 0 otherwise.