HW 6 (due on Oct 6, Friday, in the class)

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

1. Take $GL(n, \mathbb{R})$. Using the coordinates induced from $\mathbb{R}^{n^2}$, write down an example of a left-invariant vector field and of a left-invariant one-form (a form such that $L_a^*\omega = \omega$ for all $a \in G$) explicitly.

2. The following exercise proves that $SU(2)$ is the double cover of $SO(3)$. Firstly, define the Quaternions $\mathbb{H} = \mathbb{R}^4$ to be numbers of the form $a = x+yi+zj+wk$ where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji, jk = -kj = i, ki = ik = -j$. (Actually the last three conditions are somewhat redundant and can be replaced with $ijk = -1$.) We can now multiply and add quaternions. Multiplication is not commutative but is associative. Define the norm of a quaternion $\|a\| = \sqrt{x^2 + y^2 + z^2 + w^2}$ and the conjugate $\bar{x} = a - bi - cj - dk$. Prove the following. Also, the real part of a quaternion a is x.

   (a) $\|a\|^2 = a\bar{a} = \bar{a}a$.

   (b) $\|ab\| = \|a\|\|b\|$.

   (c) The set S of unit quaternions (i.e. satisfying $\|a\| = 1$) with the topology induced from $\mathbb{R}^4$ is a Lie group.

   (d) $SU(2)$ is homomorphic as a Lie group to S.

   (e) The map $f_q : S \to S$ given by $f_q(a) = qa\bar{q}$ can be identified with an element of $SO(3)$. Thus there is a map $g : q \to f_q$ from $SU(2) \to SO(3)$.

   (f) Prove that g is smooth and that g_* at the identity is an isomorphism.

   (g) Using the above prove that g_* is an isomorphism of tangent spaces everywhere.

   (h) (Optional, will not be graded) There are only two unit quaternions q such that $f_q = Id$.

   (i) (Optional, will not be graded) Prove that g is surjective onto $SO(3)$.

   (j) Using the above, prove that g is a 2-sheeted covering map.