NOTES FOR 11 JAN (THURSDAY)

1. Recap

(1) Proved some properties of mollifiers (Evans' appendix).
(2) Defined the Sobolev norm and the Sobolev space. Proved that the Sobolev space is a Hilbert space and that smooth functions are dense in it.
(3) Proved the Sobolev embedding theorem. Defined compact operators between Banach spaces.

2. Weak solutions and Sobolev spaces

Theorem 2.1. *The following inclusions are compact. (Sometimes, this along with the above theorem are referred to as the Sobolev embedding theorems.)*

(1) $H^s \subset H^l$ *if* $l < s$. *(Rellich lemma.)*
(2) $H^s \subset C^a(S^1 \times S^1 \ldots)$ *if* $s \geq [\frac{n}{2}] + a + 1$ *where* C^a *is the space of* C^a *functions with the norm* $\|f\| = \max_{S^1 \times S^1 \ldots} |f(x)| + \max |Df| + \ldots + \max |D^a f|$. *(Rellich-Kondrachov compactness.)*
(3) *Suppose* U *is a bounded domain in* $\mathbb{R}^n$, *then* $C^{k,\alpha}(\bar{U}) \subset C^{k,\beta}(\bar{U})$ *if* $\beta < \alpha$ *and* $C^k \subset C^l$ *if* $l < k$. *(The Hölder space* $C^{k,\alpha}(\bar{U})$ *consists of* $C^{k,\alpha}$ *functions with the norm* $\|f\| = \max_{\bar{U}} |f| + \max |Df| + \ldots + \max |D^k f| + \sum_{|\alpha|=k} \sup_{x,y \in \bar{U}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\alpha}$. *This space is a Banach space.)*

Proof. (1) If f_n is a bounded sequence in H^s, then $|\hat{f}_n(\vec{k})|^2(1+|k|^2)^s$ is a bounded sequence of real numbers for all k. Enumerate $\vec{k}$ by positive integers a. Therefore, by completeness of reals, we may assume that there exists a subsequence of functions $a_{1i}(x) = f_{n_i}(x)$ such that $\hat{a}_{1i}(1)^2(1+|k|^2)^s$ converges to a real number. From this subsequence choose a further subsequence $a_{2i}(x)$ such that $\hat{a}_{2i}(1)(1+|k_1|^2)^s$, $\hat{a}_{2i}(2)(1+|k_2|^2)^s$ converge. Continue like this. Now choose the diagonal subsequence $b_i(x) = a_{ii}(x)$. It is easy to see that $\hat{b}_i(\vec{k})(1+|k_i|^2)^s$ is Cauchy for all $\vec{k}$.
Now, $\|b_i - b_j\|^2_{H^l} = \sum_{\vec{k}} |\hat{b}_i(\vec{k}) - \hat{b}_j(\vec{k})|^2(1+|k|^2)^{l/2}$. When $|k| > N = \epsilon^{2/(l-s)}$, we see that $\sum_{|k|>N} |\hat{b}_i(\vec{k}) - \hat{b}_j(\vec{k})|^2(1+|k|^2)^{s/2} \frac{1}{(1+|k|^2)^{s/2-l/2}} \leq \frac{C}{N^{s/2-l/2}} < C\epsilon$. For the other smaller values of $|k|$, choose M is so large that that $b_i(k) - b_j(k)$ is small for all $|k| < N$ and $i, j > M$.
(2) As above, choose the subsequence $b_i(x)$. We will prove that it is Cauchy in the space C^a. If the Fourier series of $b_i - b_j$ (and its derivatives upto order a) converged to it (respectively to its derivatives) uniformly, then,

$$\|b_i - b_j\|_{C^a} = \|\sum \widehat{b_i - b_j}(\vec{k}) e^{i\vec{k}.\vec{x}}\|_{C^a} \leq \sum_{p=0}^{p=a} \sum_{\vec{k}} |\widehat{b_i - b_j}(\vec{k})||k|^p \tag{2.1}$$

As before, for $|\vec{k}| > N$, $\sum_{p=0}^{p=a} \sum_{|\vec{k}|>N} |\widehat{b_i - b_j}(\vec{k})||k|^p \leq C\|b_i - b_j\|_{H^s} \sum_{|k|>N} (1+|k|^2)^{a-s} < \epsilon$ for some large N. For $|\vec{k}| \leq N$, as before, we can choose M so that $i, j > M$ implies that the finitely many terms are small.

Now, by the Weierstrass M-test, indeed the Fourier series of $b_i - b_j$ converges uniformly to it (and likewise for its derivatives). So the above argument shows that $\|b_i - b_j\|_{C^a} < \epsilon$ if $i, j > N$.

(3) We prove that $C^{0,\alpha}$ is compactly contained in $C^{0,\beta}$ when $\beta < \alpha$. The other proofs are similar. Indeed, if f_k is a bounded sequence in $C^{0,\alpha}$, then if we prove that these functions are uniformly equicontinuous, the Arzela-Ascoli theorem extracts a uniformly convergent subsequence converging to $f \in C(\bar{U})$ out of them. Relabel this subsequence and call it f_k. Now, $|f_k(x) - f_k(y)| \leq C|x-y|^{\alpha} < \epsilon$ when $|x-y| < (\frac{\epsilon}{C})^{1/\alpha}$. Now we prove that actually $f \in C^{0,\beta}$ and that the convergence also happens in $C^{0,\beta}$. Indeed, $\frac{|f(x)-f(y)|}{|x-y|^{\beta}} \leq \frac{|f_k(x)-f_k(y)|}{|x-y|^{\beta}} + \frac{|f(x)-f_k(x)|}{|x-y|^{\beta}} + \frac{|f(y)-f_k(y)|}{|x-y|^{\beta}}$. Choose k (depending on x, y) to be so large that $\frac{|f(x)-f_k(x)|}{|x-y|^{\beta}} < 1$ and $\frac{|f(y)-f_k(y)|}{|x-y|^{\beta}} < 1$. Now the first term is of course bounded independent of x, y (because $\alpha > \beta$). Hence $f \in C^{0,\beta}$.

Now $\frac{|(f_m(x)-f_n(x))-(f_m(y)-f_n(y))|}{|x-y|^{\beta}} \leq C|x-y|^{\alpha-\beta}$. So if $|x-y|$ is small, this is small. If not, then the numerator is small by uniform convergence. Indeed,

$$\frac{|(f_m(x)-f_n(x))-(f_m(y)-f_n(y))|}{|x-y|^{\beta}} \leq C|x-y|^{\alpha-\beta} < \epsilon \tag{2.2}$$

if $|x-y| < \left(\frac{\epsilon}{C}\right)^{1/(\alpha-\beta)}$. If $|x-y| \geq \left(\frac{\epsilon}{C}\right)^{1/(\alpha-\beta)}$, then

$$\frac{|(f_m(x)-f_n(x))-(f_m(y)-f_n(y))|}{|x-y|^{\beta}} \leq \left(|(f_m(x)-f_n(x))| + |(f_m(y)-f_n(y))|\right)\left(\frac{\epsilon}{C}\right)^{-1/(\alpha-\beta)} < \epsilon \tag{2.3}$$

if $n, m > N$ where N is independent of x, y because f_n is Cauchy in C^0. Hence $f_k \to f$ in $C^{0,\beta}$ by the completeness of these Hölder spaces. (HW problem)

□

3. Constant-coefficient elliptic operators on the torus

Everything we did earlier holds true for vector-valued periodic functions, i.e., $\vec{u} : S^1 \times \ldots S^1 \to \mathbb{R}^{\mu}$. (By the way, these things work even when $\mathbb{R}$ is replaced by $\mathbb{C}$ on the right hand side, i.e., for complex-valued functions.) We can define a Fourier series if $\vec{u} \in L^1_{loc}$, $\widehat{\vec{u}(\vec{k})} = \frac{1}{(2\pi)^n}\int\int \ldots \vec{u}(\vec{x})e^{-i\vec{k}.\vec{x}}d^n x$. We can define Sobolev spaces $H^s(S^1 \times S^1 \ldots, \mathbb{R}^{\mu})$, and prove the Sobolev embedding and compactness theorems. The Parseval-Plancherel theorem also holds. Moreover, so does the formula relating the Fourier transform of the derivative to that of the function. (By the way, $\langle \vec{u}, \vec{v} \rangle = \sum (1+|k|^2)^s \widehat{\vec{u}}.\widehat{\vec{v}}$.)

Instead of studying $\Delta \vec{u} = \vec{f}$, let us generalise much more. Suppose we want to study

$$L(\vec{u}) = \sum_{|\alpha|=l} [A]_{l,\alpha} D^{\alpha}\vec{u} + \sum_{|\alpha|=l-1} [A]_{l-1,\alpha} D^{\alpha}\vec{u} + \ldots = \vec{f},$$

where $A_{k,\alpha}$ are $\mu \times \mu$ matrices of constants, one for each l, α such that $|\alpha| = \alpha_1 + \alpha_2 + \ldots = l$.

Now $L : H^{s+l} \to H^s$ is a bounded linear map. Take Fourier (series) transform on both sides. Now

$$\left(\sum_{|\alpha|=l} [A]_{l,\alpha}(ik)^{\alpha} + \sum_{|\alpha|=l-1} [A]_{l-1,\alpha}(ik)^{\alpha} + \ldots\right)\widehat{\vec{u}}(\vec{k}) = \widehat{\vec{f}}(\vec{k}).$$

This means that for large $|\vec{k}|$, the equation above has a solution if and only if the top order term is invertible, i.e., $\sigma_{\vec{k}} = \sum_{|\alpha|=l} [A]_{l,\alpha}(ik)^{\alpha}$ is an invertible $\mu \times \mu$ matrix for all $|k| \neq 0$. (Note that by homogenity, if it is invertible for all large $|k|$, then it is so for all non-zero ones.)

Definition 3.1. A linear differential operator L with constant coefficients on the torus is said to be elliptic if the principal symbol $\sigma_{\vec{k}}$ is invertible for all $|k| \neq 0$.

Assume that L is elliptic. Because the A are constants, there exist constants (called the ellipticity constants) δ_1, δ_2 such that $\delta_2 \|\vec{k}\|^l \|\vec{v}\| \geq \|[\sigma_{\vec{k}}][\vec{v}]\| \geq \delta_1 \|\vec{k}\|^l \|\vec{v}\|$ for all $\mu \times 1$ column vectors $\vec{v}$.