NOTES FOR 13 FEB (TUESDAY)

1. Recap

(1) Defined the first Chern class of a complex line bundle.
(2) Defined connections on direct sums, tensor products, duals, and alternating tensors.
(3) Defined higher order derivatives and defined PDE on manifolds, and gave examples.
(4) Defined the Levi-Civita connection and proved it is well-defined. Gave a physical interpretation of torsion (torsion connections spin stuff when you parallel transport).

2. Connections and curvature

Actually, there is another way of looking at the torsion-free condition.

Theorem 2.1. *Suppose M is a manifold. Let ∇^* be the induced connection on T^*M from any connection on TM. Then $d^{\nabla^*} : \Omega^1(M) \to \Omega^2(M)$.*

(1) *$(d^{\nabla^*} - d)\omega$ satisfies tensoriality and hence there exists a tensor $T \in \Gamma(T^{**}M \simeq TM \otimes \Omega^2(M))$ such that $T_\omega(\_,\_) = (d^{\nabla^*} - d)(\omega)$.*
(2) *$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. Thus for the Levi-Civita connection, $d^{\nabla^*} = d$.*

Proof. (1) $(d^{\nabla^*} - d)(f\omega) = df \wedge \omega + f d^{\nabla^*}\omega - df \wedge \omega - f d\omega = f(d^{\nabla^*} - d)\omega$. Hence, by tensoriality there exists such a tensor T (T is called the Torsion tensor of ∇).

(2) Suppose

$$\begin{aligned}
T_\omega(X,Y) &= T_\omega(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}) = X^i Y^j T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = X^i Y^j (d^{\nabla^*} - d)(\omega)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\
&= X^i Y^j (d^{\nabla^*} - d)(\omega_k dx^k)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = X^i Y^j (A^*)^a_{_k} \wedge \omega_a dx^k((\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})) \\
&= X^i Y^j (\delta^k_j (A^*)^a_{_k}(\frac{\partial}{\partial x^i})\omega_a - \delta^k_i (A^*)^a_{_k}(\frac{\partial}{\partial x^j})\omega_a) = X^i Y^j (\omega_a \left(-A^{_a}_j(\frac{\partial}{\partial x^i}) + A^{_a}_i(\frac{\partial}{\partial x^j})\right)) \\
&= \omega(\nabla_X Y - \nabla_Y X - [X,Y])
\end{aligned} \tag{2.1}$$

$\square$

Now we calculate the curvature F of the Levi-Civita connection. Note that $F^i_{\_j} = dA^i_{\_j} + A^i_{\_k} \wedge A^k_{\_j}$ locally. Now $A^i_{\_j} = \Gamma^i_{j\mu} dx^\mu$. The formula for the Christoffel symbols is $\Gamma^i_{j\mu} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mj}}{\partial x^\mu} + \frac{\partial g_{m\mu}}{\partial x^j} - \frac{\partial g_{j\mu}}{\partial x^m}\right)$. Now $F^i_{\_j} = F^i_{\_j\mu\nu} dx^\mu \wedge dx^\nu$

$$\begin{aligned}
F^i_{_j\mu\nu} &= \frac{\partial \Gamma^i_{j\nu}}{\partial x^\mu} - \frac{\partial \Gamma^i_{j\mu}}{\partial x^\nu} + \Gamma^i_{k\mu}\Gamma^k_{j\nu} - \Gamma^i_{k\nu}\Gamma^k_{j\mu} \\
&= \textit{Random nonsense involving two derivatives of the metric}
\end{aligned} \tag{2.2}$$

Typically, instead of writing F for the curvature, we use the letter R (for Riemann). The Riemann curvature tensor is a $(1,3)$ tensor $R^i_{\_j\mu\nu}$ given by the above formula. It is antisymmetric in the first two and the last two indices. It also satisfies the following two identities (the first of which is called Bianchi's first identity).

$$
\begin{aligned}
&R^i_{\ j\mu\nu} + R^i_{\ \mu\nu j} + R^i_{\ \nu j\mu} = 0 \\
&R^i_{\ j\mu\nu} = R^\mu_{\ \nu ij} \qquad (2.3)
\end{aligned}
$$

For any connection on any bundle, $dF = d(dA) + d(A \wedge A) = 0 + dA \wedge A - A \wedge dA = (dA + A \wedge A) \wedge A - A \wedge (dA + A \wedge A) = F \wedge A - A \wedge F = [F, A]$. This is called Bianchi's second identity. So of course it holds for Levi-Civita connection in particular.

There is another way to view the Riemann curvature tensor. Note that

$$FZ = (d^\nabla)^2 Z \Rightarrow R(X,Y)Z = (d^\nabla)^2 Z(X,Y) = (d^\nabla \nabla Z)(X,Y)$$

Now suppose $\nabla Z = \sum_{ij} \omega_i \otimes S_j$. Then

$$
\begin{aligned}
R(X,Y)Z = d^\nabla \nabla Z(X,Y) &= \sum (d\omega_i(X,Y)S_j - \omega_i(X)\nabla_Y S_j + \omega_i(Y)\nabla_X S_j) \\
&= \sum (X(\omega_i(Y)) - Y(\omega_i(X)) - \omega_i([X,Y])S_j - \omega_i(X)\nabla_Y S_j + \omega_i(Y)\nabla_X S_j) \\
&= \sum (\nabla_X(\omega_i(Y)S_j) - \nabla_Y(\omega_i(X)S_j) - \omega_i([X,Y])S_j - \omega_i(Y)\nabla_X S_j \\
&\quad + \omega_i(X)\nabla_Y(S_j) - \omega_i(X)\nabla_Y S_j + \omega_i(Y)\nabla_X S_j) \\
&= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad (2.4)
\end{aligned}
$$

So the curvature matrix of 2-forms is $F^i_{\ j} = dx^i(R(,)\frac{\partial}{\partial x^j})$.

We can contract the Riemann curvature tensor in a number of ways (because it is a $(1,3)$-tensor). They are all equivalent upto a sign. The chosen one is the

Definition 2.2. The Ricci tensor is a $(0,2)$-tensor defined as $Ric(Y,Z) = tr(X \to R(X,Y)Z)$, i.e., locally, $Ric_{ab} = R^c_{\ bca} = R^c_{\ acb} = Ric_{ba}$. So the Ricci tensor is symmetric.

The Ricci curvature is very important. Here is a beautiful theorem (Bonnet-Myers) that illustrates its importance.

Theorem 2.3. *If $k > 0$, $m = dim(M)$, and g is a complete Riemannian metric satisfying $Ric(p) - (m-1)kg(p) \geq 0 \forall\ p \in M$ (as semi-positive definite matrices), then $diam(M) \leq \frac{\pi}{\sqrt{k}}$. Hence M is compact. Moreover, since the universal cover is also compact (by pulling back the metric), the fundamental group is finite.*

Actually, something stronger holds (Cheng's rigidity) : If equality holds for the diameter, then (M, g) is isometric to a Sphere of an appropriate radius (depending on k). Now, because the Ricci tensor is symmetric,

Definition 2.4. A Riemannian manifold (M, g) is said to be Einstein if $Ric = \lambda g$ where λ is a constant.

The terminology comes from general relativity. (In a spacetime consisting of vaccuum, Einstein's equations with a cosmological constant are $Ric = \lambda g$ where g is a Lorentzian metric.) You can't always find Einstein metrics on every manifold. There are topological obstructions.

We can further contract the Ricci tensor to get a single function representing some aspect of curvature.

Definition 2.5. The scalar curvature is a function $S = Ric_{ab} g^{ab}$.

The scalar curvature can be interpreted as follows :

$$\frac{Vol(B_\epsilon(p) \subset M}{B_\epsilon(0) \subset \mathbb{R}^m} = 1 - \frac{S}{6(m+2)}\epsilon^2 + O(\epsilon^4). \tag{2.5}$$

Therefore, if $S > 0$, then balls in the manifold have smaller volume because they curve more.

The Yamabe problem asks the following : On a compact manifold, given a metric g_0, is there is a function f so that $g = e^f g_0$ has constant scalar curvature ? (The answer is now known to be "yes"). If the "constant" of the scalar curvature is negative, then it is not incredibly hard to prove the theorem. There are obstructions to finding metrics of positive scalar curvature on manifolds (you can't always do it). Even if you can find one, the Yamabe problem in the positive case is very hard. Shockingly enough, its proof involves the positive mass theorem of general relativity.