NOTES FOR 13 MAR (TUESDAY)

1. Recap

(1) Stated the Hodge theorem.
(2) Applied it to prove Poincarè duality and a sketch of the proof of the Kunneth formula.

2. Statement of the Hodge theorem and applications

For the flat torus, since we already proved that Δ_d is a constant coefficient symmetric elliptic operator, and that elliptic operators are Fredholm, we see that $\Delta_d \eta = \omega$ can be solved for η if and only if ω is orthogonal to the space of harmonic forms (which we proved is finite dimensional). Moreover, we can choose η to be the unique one having the smallest L^2-norm. So we have $\eta = G(\omega)$. Thus, every form ω can be uniquely written as $H(\omega) + \Delta G(\omega)$. Now $\Delta_d d(G\omega) = dd^\dagger dG\omega = d\Delta_d G\omega = d\omega$. This does not yet show that $d(G\omega) = G(d\omega)$. We need to show that $d(G\omega)$ has the smallest L^2-norm among all such solutions, i.e., it is orthogonal to harmonic forms. But indeed, $(d(G\omega), \alpha) = (G\omega, d^\dagger \alpha) = 0$. Likewise, G commutes with $d^\dagger$. As for the completeness of the eigenfunctions, one can explicitly calculate these eigenfunctions as simply being of the form e^{ikx}. We know that the Fourier functions are complete in L^2 (Parseval-Plancherel).

Seeing how useful this Hodge theorem is, we want to prove it for general compact oriented (M, g). There are several approaches to this. One is to prove such a result for general elliptic operators. (However, that approach has the disadvantage that it does not say much about eigenvalues. So we have to deal with that issue.)

3. Sobolev spaces on general manifolds

The theory of Sobolev spaces, Sobolev embedding, etc goes over to general manifolds. We will focus on that now.

There are many ways of defining $H^s(M, E)$:

Definition 3.1. Suppose (E, h, ∇) is a vector bundle with metric and connection on a compact oriented (M, g) and $s \geq 0$ is an integer. Suppose t is a smooth section of E. Define $\|t\|_{H^s}^2 = \int_M (|t|^2 + |\nabla t|^2 + \ldots + |\nabla^s t|^2) vol_g$. Define $H^s_{\nabla,h,g}$ to be the completion of this space (in the metric space sense). Concretely, H^s consists of L^2 sections t such that there exist smooth sections $t_n \to t$ in L^2 and t_n form a Cauchy sequence in the H^s norm.

The claim is that these spaces are equivalent. Indeed,

Lemma 3.2. *The Sobolev norms are equivalent (on smooth sections) for different h, ∇, g.*

Proof. Suppose we choose $h_1, \nabla_1, g_1,\ h_2, \nabla_2, g_2$. First of all, just as in the exam, it is easy to see that there exists a positive finite constant C so that $\frac{1}{C} h_1 \leq h_2 \leq C h_2$, $\frac{1}{C} g_2 \leq g_1 \leq C g_2$ where the inequalities are in the sense of positive-definite matrices. Now $\nabla_1 = \nabla_2 + B$ where B is an endomorphism of E. Let $|B|_1, |B|_2 \leq C$. Now $\frac{1}{C^2} |\nabla_1 t|_{h_2 \otimes g_2} \leq |\nabla_1 t|^2_{h_1 \otimes g_1} \leq C^2 |\nabla_1 t|^2_{h_2 \otimes g_2}$. Now $|\nabla_1 t|_{h_2 \otimes g_2} \leq |\nabla_2 t|_2 + C|t|_2$. Moreover, $|\nabla_2 t|_2 \leq |\nabla_1 t|_2 + C|t|_2$. Hence, $\frac{1}{K}(|t|_2^2 + |\nabla_2 t|_2^2) \leq |t|_2^2 + |\nabla_1 t|_2^2 \leq K(|t|_2^2 + |\nabla_2 t|_2^2)$. By induction, we can show this for all derivatives. ∎

Remark 3.3. Note that the above proof works even for open subsets U of a compact manifold M.

To make another definition, we need a lemma :

Lemma 3.4. *If $\vec{s} : U \subset \mathbb{R}^m \to \mathbb{R}^r$ is in L^1_{loc} and weakly differentiable with weak derivatives $\partial_i \vec{s} = \vec{t}_i$, then for any smooth functions $g : U \to GL(r, \mathbb{R})$, diffeomorphisms $y(x) : U \to U$, the function $\vec{\tilde{s}} = g\vec{s}$ is weakly differentiable with weak derivative $\frac{\partial \vec{\tilde{s}}}{\partial y^i} = \frac{\partial g(x(y))}{\partial y^i} g^{-1} \vec{\tilde{s}} + g\vec{t}_j \frac{\partial x^j}{\partial y^j}$. (Note that this coincides with what we expect if $\vec{s}$ is smooth.)*

Proof. Indeed, if $\vec{\phi}$ is a smooth function with compact support in U, then

$$\int_U \Big(\langle \frac{\partial g(x(y))}{\partial y^i} g^{-1} \vec{\tilde{s}} + g\vec{t}_j \frac{\partial x^j}{\partial y^j}, \phi \rangle dy = \int_U \langle \frac{\partial g(x(y))}{\partial y^i} g^{-1} \vec{\tilde{s}}, \phi \rangle + \frac{\partial x^j}{\partial y^j} \langle \vec{t}_j, g^T \phi \rangle \Big) dy$$

$$= \int_U \langle \vec{\tilde{s}}, (\frac{\partial g(x(y))}{\partial y^i} g^{-1})^T \phi \rangle dy - \int_U \langle \vec{\tilde{s}}, (g^{-1})^T \frac{\partial}{\partial x^j} \left(\sqrt{\det\left(\frac{\partial \vec{y}}{\partial \vec{x}}\right)} \frac{\partial x^j}{\partial y^j} g^T \phi \right) \rangle \sqrt{\det\left(\frac{\partial \vec{x}}{\partial \vec{y}}\right)} dy$$

$$= -\int_U \langle \vec{\tilde{s}}, \frac{\partial \phi}{\partial y^i} \rangle dy - \int_U \langle \vec{\tilde{s}}, \frac{\partial}{\partial x^j} \left(\sqrt{\det\left(\frac{\partial \vec{y}}{\partial \vec{x}}\right)} \frac{\partial x^j}{\partial y^j} \right) \phi \rangle \sqrt{\det\left(\frac{\partial \vec{x}}{\partial \vec{y}}\right)} dy = -\int_U \langle \vec{\tilde{s}}, \frac{\partial \phi}{\partial y^i} \rangle dy \tag{3.1}$$

□

This shows that the notion of weak differentiability of an L^1_{loc} section of a vector bundle is well-defined in terms of coordinates and trivialisations.

Lemma 3.5. *Suppose (E, ∇, h) is a bundle with a metric and a compatible connection on (M, g) where M is any orientable manifold (not necessarily compact). Let $s \in L^1_{loc}(M)$ be a weakly differentiable section. Then the weak derivative ∇s is well-defined as an L^1_{loc} section of $T^*M \otimes E$ and satisfies $(\nabla s, \phi)_{L^2} = (s, \nabla^\dagger \phi)_{L^2}$ where ϕ is any compactly supported smooth section on M and $\nabla^\dagger$ is given by the same formula as before. Conversely, if this property is satisfied, then s is weakly differentiable (in the sense defined before).*

Proof. Define ∇s locally as $\frac{\partial \vec{s_\alpha}}{\partial x^i} dx^i + A_\alpha \vec{s}_\alpha$ where the derivatives are weak derivatives. From the previous lemma it is easily seen that it transforms like a section of $T^*M \otimes E$.

Suppose we cover M by a locally-finite cover U_α of charts which are also trivialising neighbourhoods, and we let ρ_β be a partition-of-unity subordinate to it (Note that ρ_β has compact support in some U_β but the indexing set need not be the same.) Then $(\nabla s, \phi) = \sum_\beta (\nabla s, \rho_\beta \phi)$ (the sum is finite because ϕ has compact support). Now $(\nabla s, \phi) = -\sum_\beta (s, d^\dagger(\rho_\beta \phi)) + \sum_\beta (s, A^\dagger \rho_\beta \phi) = -\sum_\beta (s, \nabla^\dagger(\rho_\beta \phi)) = -\sum_\beta (s, \nabla^\dagger \phi)$ (where we used the property that $\nabla^\dagger$ is a first order differential operator and $d(\sum \rho_\beta) = 0$).

The converse part follows by taking ϕ to be supported in a coordinate trivialising open set. □

Now we define the Sobolev space in another way.

Definition 3.6. Suppose (E, ∇, h) is a bundle with a metric and a compatible connection on a compact oriented (M, g). Let $s \geq 0$ be an integer. Then the space $\tilde{H}^s_{\nabla,h,g}$ consists of s times weakly differentiable sections $\in L^2$ with inner product $(a, b) = \displaystyle\int \langle a, b \rangle vol_g + \langle \nabla a, \nabla b \rangle vol_g + \ldots$ where the derivatives are weak derivatives.

Lemma 3.7. *$\tilde{H}^s_{\nabla,h,g}$ is a Hilbert space and smooth sections are dense in it. Hence it coincides with $H^s_{\nabla,h,g}$.*

Proof. Hilbert space : If f_n is a Cauchy sequence, then ρf_n is also a Cauchy sequence for any smooth function ρ. Assume that ρ is compactly supported in a coordinate trivialising neighbourhood U. Thus ρf_n can be extended smoothly to $S^1 \times S^1 \ldots$ (by simply taking a large cube in $\mathbb{R}^m$ containing its support and periodically extending it). Moreover, it is also clear that ρf_n is Cauchy in $H^s(S^1 \times S^1 \ldots)$. Hence, $\rho f_n \to u$ for some $u \in H^s(S^1 \times S^1 \ldots)$. This function u has support in the previously chosen large rectangle and hence can be extended to all of M. Moreover, since the Sobolev norms are equivalent, this convergence happens in $H^s_{\nabla,h,g}$. $f_n = \sum \rho_\alpha f_n \to \sum u_\alpha$ in H^s where ρ_α is a partition-of-unity.
Smooth functions are dense : Suppose $\rho_\alpha \geq 0$ is such that $\sum \rho_\alpha^2 = 1$ and these are subordinate to a finite trivialising coordinate cover U_α. Suppose $f \in H^s_{\nabla,h,g}$. Then there are sequences of smooth functions $f_{n,\alpha} \to \rho_\alpha f$ in $H^s(S^1 \times S^1 \ldots)$. Now $\rho_\alpha f_{n,\alpha}$ is well-defined on M. Moreover, $\| \sum \rho_\alpha f_{n,\alpha} - \rho_\alpha \rho_\alpha f \|_{H^s_{\nabla,h,g}} \leq C \sum_\alpha \| \sum f_{n,\alpha} - \rho_\alpha \rho_\alpha f \|_{H^s(S^1 \times S^1 \ldots)} \to 0$. ☐

There is yet another way to define the Sobolev space.

Definition 3.8. Choose a finite cover of trivialising coordinate neighbourhoods $(U_\alpha, x^i_\alpha, e_{j,\alpha})$ and a partition-of-unity subordinate to it. The space $H^{`s}$ is the space of all L^1_{loc} sections a such that $\|a\|^2 = \|\rho_\alpha \vec{a}_\alpha\|_{H^s(S^1 \times S^1 \ldots)} < \infty$. The inner product between a and b is $\displaystyle\sum_\alpha (\rho_{al} \vec{a}_\alpha, \rho_\alpha \vec{b}_\alpha)_{H^s}$

Lemma 3.9. *The space $H^{`s}$ is well-defined independent of choices. It is a Hilbert space and smooth sections are dense in it. On smooth functions the $H^{`s}$ norm is equivalent to the $H^s_{\nabla,h,g}$ norm with respect to any connection and hence it is homeomorphically isomorphic to $H^s_{\nabla,h,g}$.*

The proof is very similar to the earlier ones and is left as an exercise.

Remark 3.10. One can define the $W^{k,p}$ spaces and the $C^{k,\alpha}$ spaces too.