NOTES FOR 15 FEB (THURSDAY)

1. Recap

(1) Defined the torsion tensor (as a natural obstruction) and proved the LC connection has zero torsion tensor.
(2) Wrote formulae for the Riemann curvature tensor and described its symmetries.
(3) Defined the Ricci tensor and the scalar curvature. Defined Einstein metrics. Stated the Bonnet-Myers and Cheng rigidity theorems. Also stated the Yamabe problem.

2. Connections and curvature

One-manifolds do not have any curvature. On a surface the scalar curvature determines the full Riemann curvature tensor. Indeed, $R_{abcd} = -g_{ai}R^i_{bcd} = K(g_{ac}g_{db} - g_{ad}g_{cb})$ where $2K = S$.

On a three manifold, the Ricci curvature determines it. From four manifolds onwards, these are all different.

Note that if we are given an orthonormal X, Y in T_pM, then $\exp(uX + vY)$ is locally a surface in the manifold. It has the induced metric and the induced curvature (all curvatures are the same on a surface). This curvature turns out to be $K(X,Y) = g(R(X,Y)Y,X)$. These curvatures (for every two orthonormal X, Y) are called sectional curvatures of 2-planes. More generally, the sectional curvature is defined as $K(X,Y) = \frac{g(R(X,Y)Y,X)}{(X,X)(Y,Y)-(X,Y)^2}$. Define $\kappa(X,Y) = g(R(X,Y)Y,X)$. It turns out (an easy calculation) that the sectional curvatures completely determine the Riemann curvature tensor.

$$\begin{aligned}
6g(R(x,y)z,w) &= \kappa(x+w,y+z) - \kappa(x,y+z) - \kappa(w,y+z) - \kappa(y+w,x+z) \\
&\quad +\kappa(y,x+z) + \kappa(w,x+z) - \kappa(x+w,y) - \kappa(x+w,z) \\
&\quad +\kappa(x,y) + \kappa(w,y) + \kappa(x,z) + \kappa(w,z) \\
&\quad +\kappa(y+w,x) + \kappa(y+w,z) - \kappa(y,x) - \kappa(w,x) - \kappa(y,z) - \kappa(w,z)
\end{aligned} \tag{2.1}$$

A manifold is said to have positive sectional curvature if all the sectional curvatures are positive (at all points and all 2-planes) and likewise for negative sectional curvature, etc. Just as the metric has the same symmetries as the Ricci tensor (and hence it makes sense to ask for Einstein metrics, which are essentially "constant" Ricci curvature), the Riemann curvature tensor has the same symmetries as $g_{ac}g_{db} - g_{ad}g_{cb}$. It turns out that

Lemma 2.1. *The sectional curvatures of all 2-planes are the same everywhere if and only if* $-g_{ai}R^i_{bcd} = R_{abcd} = K(g_{ac}g_{db} - g_{ad}g_{cb})$ *where* K *is a constant (equal to the sectional curvatures). Such a metric is said to have constant sectional curvature.*

Now we shall write some examples :

(1) $\mathbb{R}^n, Euc$ obviously has zero curvature.
(2) The torus $S^1 \times S^1 \ldots, d\theta^1 \otimes d\theta^1 + \ldots$ also has 0 curvature (locally it is the same as $\mathbb{R}^n$).
(3) The induced metric on $S^n \subset \mathbb{R}^{n+1}$ is preserved under $SO(n+1)$ (because the Euclidean metric is so and S^n itself is preserved). It is easy to see that if $f : M \to N$ is an isometry, then $f^*Riemann_N = Riemann_M$. Therefore, the sectional curvatures are the same at every

point. The constant K is positive because it is so at the north pole (where it can be calculated easily).

(4) The product metric on $S^2 \times S^2$ has non-negative sectional curvature (but some 2-planes have 0 sectional curvature). It is an open problem (The Hopf conjecture) as to whether there is a metric having strictly positive sectional curvature on $S^2 \times S^2$.

(5) The Hyperbolic metric on $\mathbb{H}^n$ is $g = \frac{\sum (dx^i)\otimes(dx^i)}{(x^n)^2}$. It is "conformal" to the Euclidean metric (i.e., it is the Euclidean metric times a function). The curvature can be easily calculated knowing how to compute the curvature of $g = fg_0$. The Christoffel symbols are

$$\begin{aligned} \Gamma^i_{jk} &= \frac{(x^n)^2}{2}\left(\frac{\partial f}{\partial x^\mu}\delta^i_j + \frac{\partial f}{\partial x^j}\delta^i_\mu - \frac{\partial f}{\partial x^i}\delta_{j\mu}\right) \\ &= -\frac{1}{x^n}(\delta_{n\mu}\delta^i_j + \delta_{nj}\delta^i_\mu - \delta_{ni}\delta_{j\mu}) \end{aligned} \tag{2.2}$$

Now one easily calculate the Riemann curvature tensor to prove that the sectional curvatures are constant equal to -1.

There are a number theorems about sectional curvature.

(1) Complete Riemannian manifolds with constant sectional curvature are an isometric quotient of space forms : Hyperbolic space, or Euclidean space, or the Sphere. (Killing-Hopf theorem)
(2) If the sectional curvature of a complete manifold is non-positive, then the universal cover is diffeomorphic to $\mathbb{R}^n$ (Cartan-Hadamard theorem).
(3) If a complete Riemannian manifold has negative sectional curvature, then every nontrivial abelian subgroup of $\pi_1(M)$ must be $\mathbb{Z}$ (Preissman's theorem).
(4) (Synge's theorem) If a complete Riemannian manifold has positive sectional curvature, then
    (a) If M is even-dimensional and orientable, then M is simply connected
    (b) If M is odd-dimensional, it is orientable.

3. Divergence, Stokes' theorem, and Laplacians

Suppose $u : M \to \mathbb{R}$ is a function on a Riemannian manifold (M, g) whose tangent bundle is equipped with the Levi-Civita connection. Then $\nabla u = \frac{\partial u}{\partial x^j} g^{ij} \frac{\partial}{\partial x^i}$ is called the gradient of u with respect to g. It is just dual to du using the metric g. Suppose c is a regular value of u, then $u^{-1}(c)$ is a submanifold of M of dimension $m-1$. The gradient ∇u is normal to this submanifold. Indeed, if $\vec{v}$ is tangent to the submanifold, i.e., $\vec{v} = \frac{d\gamma}{dt}(0)$ where γ is a curve lying on $u^{-1}(c)$, i.e., $u(\gamma(t)) = c$, then $\frac{du}{dt} = 0$, i.e., $0 = \frac{\partial u}{\partial x^i}\frac{\partial \gamma^i}{\partial t} = (\nabla u)^j g_{ij}\frac{\partial \gamma^i}{\partial t}$. Thus ∇u is perpendicular to $\vec{v}$.

Suppose X is a smooth vector field. Define the divergence of X

Definition 3.1. $div(X) = \nabla_i X^i = \frac{\partial X^i}{\partial x^i} + \Gamma^i_{ik} X^k$. So in normal coordinates, it is the usual divergence at p. Note that $div(fX) = X(f) + f div(X)$.