# NOTES FOR 15 MAR (THURSDAY)

## 1. Recap

(1) Defined Sobolev spaces in various ways and showed their equivalence (i.e., they are all quasi-isometrically isomorphic).
(2) The slogan is "Sections that are compactly supported on a trivialising coordinate neighbourhood can be thought of as sitting on a flat torus (and the Sobolev norms are equivalent)."

## 2. Sobolev embedding and compactness

Define $C^{k,\alpha}(M,E)$ as the space of $C^k$ sections of $E$ such that in local coordinates (and frames) they are $C^{k,\alpha}$. The norm on this space is $\|u\|_{C^{k,\alpha}} = \sum_{\mu} \|\vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)}$. This is independent of choices made and is a Banach space. This will be given as a HW problem.

Actually, this is equivalent to the norm $\sum \|\rho_\mu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)}$ :

*Proof.* Indeed, firstly, $\sup_x |f(x)g(x)| + \sup_{x,y} \frac{|f(x)g(x)-f(y)g(y)|}{|x-y|^\alpha} \leq \|f\|_{C^{0,\alpha}} \|g\|_{C^{0,\alpha}}$. Hence $\sum \|\rho_\mu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} \leq C\|u\|_{C^{k,\alpha}}$.

Next, if one changes coordinates and trivialisations, the resulting $C^{k,\alpha}$ norms are equivalent (a part of the the HW problem). Therefore, $\|\vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} \leq \sum_{\nu \neq \mu} \|\rho_\nu \vec{u}_\mu\| + \|\rho_\mu \vec{u}_\mu\|$. Now $\|\rho_\nu \vec{u}_\mu\|_{C^{k,\alpha}(\bar{U}_\mu)} = \|g_{\nu\mu} \rho_\nu \vec{u}_\nu\|_{C^{k,\alpha}(\bar{U})_\mu} \leq C \|\rho_\nu \vec{u}_\nu\|_{C^{k,\alpha}(\bar{U}_\nu)}$ where the last norm is in the $\nu$ coordinates. Hence we are done. □

Firstly, we have the following compactness result :

**Lemma 2.1.** *Suppose $k \leq l$. If $k < l$ or $0 \leq \beta < \alpha < 1$, then $C^{l,\alpha} \subset C^{k,\beta}$ is a compact embedding.*

*Proof.* The embedding part is trivial. We shall prove that $C^{0,\alpha} \subset C^0$ is compact (the general case is similar). Let $\rho_\alpha$ be a partition of unity. If $\|f_n\|_{C^{0,\alpha}} \leq C$, then $\|\rho_\mu f_n\|_{C^{0,\alpha}(\bar{U}_\mu)} \leq C$. By the usual Arzela-Ascoli argument as we did earlier, there is a subsequence (which we shall denote by $f_n$ still) such that $\rho_\mu f_n \to f_\mu$ on $C^{0,\alpha}(\bar{U}_\mu)$ for some function $f_\mu : U_\alpha \to \mathbb{R}^r$. (For each $\mu$ there is a potentially different subsequence. We choose one for the first $\mu$, then choose a further subsequence for the second $\mu$ and so on. There are only finitely many $\mu$.) Clearly $f_\mu$ has compact support in $U_\mu$ and hence can be extended to a $C^{0,\alpha}$ section of $E$ on $M$. Now $\|\sum_\mu f_\mu - f_n\|_{C^{0,\alpha}(M)} \leq C \sum_\mu \|f_\mu - \rho_\mu f_n\|_{C^{0,\alpha}(\bar{U}_\mu)} \to 0$. □

Now we prove Sobolev embedding plus compactness.

**Theorem 2.2.** *The following inclusions are compact. (Sometimes, this along with the above theorem are referred to as the Sobolev embedding theorems.)*

(1) *$H^s(E) \subset H^l(E)$ if $l < s$. (Rellich lemma.)*
(2) *$H^s(E) \subset C^a(M,E)$ if $s \geq [\frac{n}{2}] + a + 1$. (Rellich-Kondrachov compactness.)*

*Proof.* (1) The inclusion part is clear. If $f_n$ is a bounded sequence in $H^s(E)$, then $\rho_\alpha f_n \in H^s(S^1 \times S^1 \ldots)$ is a bounded sequence and by the usual Rellich lemma, it has a convergent subsequence (which abusing notation as usual we still denote by the subscript $n$) $\rho_\alpha f_n \to f_\alpha$

in $H^s(S^1 \times S^1 \ldots)$. Passing to a further subsequence (that converges a.e.) we see that $f_\alpha$ has support in $U_\alpha$ and hence can be thought of as being a global section on $M$. By equivalence of norms, $\rho_\alpha f_n \to f_\alpha$ in $H^s(M, E)$. Thus $\sum \rho_\alpha f_n = f_n \to \sum f_\alpha$.

(2) If $f \in H^s(E)$ then $\rho_\alpha f \in H^s(S^1 \times S^1 \ldots)$. Thus $\rho_\alpha f \in C^a(S^1 \times S^1 \ldots)$ by the usual Sobolev embedding on the torus. Hence, $\rho_\alpha f \in C^a(M, E)$ by equivalence of norms. Thus $\sum_\alpha \rho_\alpha f = f \in C^a(M, E)$. Likewise, if $f_n \in H^s(E)$ is bounded, then a subsequence $\rho_\alpha f_n \to f_\alpha$ in $C^a(S^1 \times S^1 \ldots)$. Since $f_\alpha$ is supported on $U_\alpha$, as before $f_n = \sum \rho_\alpha f_n \to \sum f_\alpha$ in $C^a(M, E)$. □

## 3. Elliptic operators - Regularity

Now we define the notion of a uniformly elliptic operator : Suppose $(E, h_E, \nabla_E), (F, h_F)$ are smooth bundles with metrics and a metric compatible connection for $E$ on a compact oriented $(M, g)$ where $TM$ is equipped with the Levi-Civita connection. Whenever we use $\nabla$ in what follows, it is made out of $\nabla_E, \nabla_g$ (Fix $h_E, h_F, \nabla_E$, and $g$ in whatever follows.)

First we prove a "structure theorem" for linear PDOs.

**Lemma 3.1.** *To every linear PDO L of order o with smooth coefficients, there exist smooth maps* $a_k : T^*M \otimes T^*M \otimes \ldots T^*M \otimes E \to F$ *(where $T^*M$ is repeated $k$ times) such that* $L(u) = \displaystyle\sum_{k=0}^{o} a_k \nabla^k u$.

*Proof.* We prove this by induction on $o$. For $o = 0$, by tensoriality, there is such an endomorphism. Assume the result for $0, 1, \ldots, o-1$. Then locally, in a trivialising coordinate chart, $L(u)_\alpha = \sum_{k=0}^{o} a^I_{k,\alpha} \partial_I \vec{u}_\alpha$. If we change the trivialising coordinate chart, then $\vec{u}_\beta = g_{\beta\alpha}\vec{u}_\alpha$, and $\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}$ (and the tensor product version of this). The highest order term changes as $a^I_{0,\alpha}\partial_{x,I}\vec{u}_\alpha \to a^I_{o,\alpha} g_{\beta\alpha} \frac{\partial y^J}{\partial x^I} \partial_{y,J}\vec{u}_\beta$, i.e., $a_o$ is a global section of $End(T^*M \otimes T^*M \ldots E, F)$. Hence $L(u) - a_o \nabla^o u$ is a linear PDO of order $o - 1$ and hence by induction we are done. □

The formal adjoint $L^*_{form}$ of $L$ is defined as being a linear PDO of the same order given by $\sum_{k=0}^{o} (\nabla^k)^\dagger \circ a_k^\dagger$. It satisfies (and is equivalent to) $(L^*_{form} u, v) = (u, Lv)$ for smooth $u, v$.

**Definition 3.2.** The principal symbol of $L$ is the Endomorphism $\sigma(L) : T^*M \otimes \ldots E \to F$ given by $\sigma(L) = a_o$.

**Definition 3.3.** A linear PDO $L$ with smooth coefficients is called uniformly elliptic with ellipticity constants $\delta_1, \delta_2 > 0$ if $\delta_1 |v|^2_{h_E(p)} \leq |\sigma_p(L)(\zeta, \zeta, \ldots, \zeta)v|^2_{h_F(p)} \leq \delta_2 |v|^2_{h_E(p)}$ $\forall\, p \in M, \zeta \neq 0 \in T^*_pM$ and the principal symbol is invertibel. (Please note that $\delta_1, \delta_2$ depend on the fixed $h_F, h_E$ obviously.) In particular, the ranks of $E$ and $F$ are required to be the same.

It is clear that $L$ is uniformly elliptic (from now on, called "elliptic") if and only if $L^*_{form}$ is so. The ellipticity constants may be chosen to be equal. (Again, the ranks of $E$ and $F$ being the same is important for this.)

**Definition 3.4.** Suppose $f$ is an $L^2$ section of $F$. An $L^2$ section $u$ is said to be a distributional solution of $Lu = f$ if for every smooth section $\phi$ of $F$, $(u, L^*_{form}\phi) = (f, \phi)$. (Please note that we have not defined distributions in general. However, the notion of a distributional solution does not need distributions.)

Next we prove that distributional solutions of elliptic equations are smooth.

**Theorem 3.5.** *If $L$ is uniformly elliptic and $f$ a smooth section of $F$. Then if $u \in L^2$ satisfies $Lu = f$ in the sense of distributions then $u$ is smooth. Moreover, if $f \in H^s$, then $u \in H^{s+l}$ and $\|u\|_{H^{s+o}} \leq C_s(\|f\|_{H^s} + \|u\|_{L^2})$ where $C_s$ depends only on $h_E, h_F, g, \nabla_E$, an upper bound on $\|a_k\|_{C^{s+o}}$, and on the ellipticity constants.*

We claim that this theorem follows from

**Theorem 3.6.** *If $L$ is uniformly elliptic, $u$ is a smooth section of $E$, then $\|u\|_{H^{s+o}} \leq C_s(\|Lu\|_{H^s} + \|u\|_{L^2})$.*

Indeed, assume this theorem. Then we shall prove theorem 3.5. Suppose $u_n$ are smooth sections converging to $u$ in $L^2$. Then $\|u_n\|_{H^{s+o}} \leq C_s(\|Lu_n\|_{H^s} + \|u_n\|_{L^2})$ according to theorem 3.5. Note that $(Lu_n, \phi) = (u_n, L^*_{form}\phi) \to (u, L^*_{form}\phi) = (f, \phi)_{L^2}\ \forall\ \phi$.