NOTES FOR 16 JAN (TUESDAY)

1. Recap

(1) Proved the compactness theorems about Sobolev and Hölder spaces.
(2) Defined constant-coefficient linear elliptic operators on the torus (the principal symbol is invertible).

2. Constant-coefficient elliptic operators on the torus

Even for elliptic operators, the above equation for Fourier coefficients cannot always be inverted. However, for sufficiently large $|k|$, it can be inverted to produce an "approximate" solution $\vec{u}_{app}$ whose Fourier coefficients are 0 for $|k| \leq N$ and $\widehat{\vec{u}_{app}}(\vec{k}) = \hat{L}_{\vec{k}}^{-1}\widehat{\vec{f}}(\vec{k})$. We claim that

Theorem 2.1. *If $\vec{f}$ is in H^s and L is elliptic, then*

(1) *$\vec{u}_{app}$ is in H^{s+l}.*
(2) *The map $G : H^s \rightarrow H^{s+l}$ given by $G(f) = \vec{u}_{app}$ is a bounded linear map (with the bound depending on the ellipticity constants, s, l, and coefficients of the lower order terms).*
(3) *$L \circ G - I : H^s \rightarrow H^s$ and $G \circ L - I : H^{s+l} \rightarrow H^{s+l}$ are compact operators. (In simple english, G is an "almost" inverse of L. It is called a parametrix for L.)*
(4) *If $\vec{u} \in H^{s+l}$ satisfies $L(\vec{u}) = \vec{f}$, then $\|u\|_{H^{s+l}} \leq C(\|f\|_{H^s} + \|u|_{L^2})$ where C depends only on the ellipticity constants, s, l, and bounds on the other coefficients (the lower order terms).*

Proof. (1) Note that $|\widehat{\vec{u}_{app}}(\vec{k})| \leq C\frac{\|\widehat{\vec{f}}(\vec{k})\|}{\|\vec{k}\|^l}$ if $|\vec{k}| \geq N$ where N is sufficiently (depending on the ellipticity constants and the coefficients of the lower order terms) large. Indeed, the magnitude of the lower order terms is less than $C(\|\vec{k}\|^{l-1} + \|\vec{k}\|^{l-2} + \ldots \leq C\|\vec{k}\|^{l-1})$ if $\|\vec{k}\| > 1$. Now $\|[\sigma_{\vec{k}} + lower][\vec{v}]\| \geq (\delta_1\|\vec{k}\|^l - C\|\vec{k}\|^{l-1})\|\vec{v}\|$. Of course if $|\vec{k}| \geq N$ is large, then $\|\hat{L}[\vec{v}]\| \geq c\|\vec{v}\|$ where $c > 0$. Hence $\|\hat{L}^{-1}[\vec{v}]\| \leq C\|\vec{k}\|^{-l}\|\vec{v}\|$ for large N.

The above easily implies that $\vec{u}_{app} \in H^{s+l}$. Moreover, $\|\vec{u}_{app}\|_{H^{s+l}} \leq C\|f\|_{H^s}$.

(2) The last inequality implies this result.
(3) $K(f) = L \circ G(f) - f = L(u_{app}) - f = -\sum_{|k|<N} \widehat{\vec{f}}(\vec{k})e^{i\vec{k}.\vec{x}}$. Now $K(f)$ is smooth and is hence in H^a $\forall$ a. By the Rellich compactness lemma, $K(f) : H^s \rightarrow H^s$ is compact. Now $G(L(u)) - u = -\sum_{|k|<N} \hat{u}(k)e^{i\vec{k}.\vec{x}}$. As before this is a smooth function and hence by the Rellich lemma, $G \circ L - I$ is compact.
(4) Taking Fourier series on both sides, $\hat{L}\widehat{\vec{u}}(\vec{k}) = \widehat{\vec{f}}(\vec{k})$. Of course, for large $|k|$, u coincides with u_{app}. For small $|k| < N$, $(1 + |k|)^{s+l} \leq (1 + N)^{s+l} \leq C$ where C depends only on N, s, l and hence only on the ellipticity constants, s, l, and the bounds on the lower order coefficients. This proves the result.

□

Now we define a useful notion in functional analysis.

Definition 2.2. Suppose H_1, H_2 are Hilbert spaces. A bounded linear operator $T : H_1 \to H_2$ is called Fredholm if $ker(T), Coker(T)$ are finite-dimensional.

We prove the following useful theorem about Fredholm operators. (These results are in https://ocw.mit.edu/courses/mathematics/18-965-geometry-of-manifolds-fall-2004/lecture-notes/lecture16_17.pdf) In these results, we use the easy fact that if T is a bounded linear operator and K is compact, then $T \circ K$ and $K \circ T$ are compact. We also use a slightly more difficult fact that if K is compact, then K^* is so as well.

Theorem 2.3.
(1) *If the range of T is closed, then $Coker(T)^* \simeq Ker(T^*)$ where $T^* : H_2^* \to H_1^*$.*
(2) *If $ker(T), Coker(T)$ are finite dimensional, then the range is closed.*
(3) *T is Fredholm if and only if T^* is so.*
(4) *T is Fredholm if and only if there exist bounded linear maps $G_1, G_2 : H_2 \to H_1$, such that $G_1 \circ T - I$, $T \circ G_2 - I$ are compact operators.*
(5) *The set of Fredholm operators $S \subset B(H_1, H_2)$ is open.*
(6) *Suppose $I \subset \mathbb{R}$ is a connected set. If $F(t) : I \subset \mathbb{R} \to S$ is a continuous map, then the index $Ind(F(t)) = dim(Ker(F(t))) - dim(Coker(F(t)))$ is a constant.*
(7) *If $K : H_1 \to H_2$ is a compact operator and T is Fredholm, then $T + K$ is Fredholm with the same index.*

Proof.
(1) Take $\rho \in ker(T^*)$ to $\lambda \in Coker(T)^*$ where $\lambda(y + TX) = \rho(y)$. This is well-defined. Since the range is closed, $Coker(T)$ is a Hilbert space (the orthogonal complement of the image). λ is a bounded linear functional on this space. Therefore, $\rho : H_2 \to Coker(T) \to \mathbb{R}$ is a bounded linear functional. This inverts the previous construction.
(2) Take $T : X = ket(T)^{\perp} \to H_2$. This is injective. Now let $C = Ran(T)^{\perp}$. Define $S : X \oplus C \to H_2$ as $S(x, c) = T(x) + c$. Now this is a bounded linear isomorphism. Hence by the open mapping theorem it is a topological isomorphism. Hence $S(X \oplus \{0\}) = Ran(T)$ is closed.
(3) If $Ker(T^*) \simeq Coker(T)^*$ and $Coker(T^*)^* = Ker(T^{**}) = Ker(T)$. This gives the result.
(4) If T is Fredholm, then $T : ker(T) \oplus ker(T)^{\perp} \to Coker(T) \oplus Im(T)$ is bounded linear and defines an injective map $T_1 : ker(T)^{\perp} \to H_2$. Define $G(a \oplus b) = T_1^{-1}(b)$. Clearly, $G \circ T - I$ is a projection onto a finite dimensional subspace and hence compact. Now $T \circ G(a \oplus b) - a \oplus b = T(T_1^{-1}(b)) - a \oplus b = -a \oplus 0$ which is another projection and hence compact.
Conversely, if there exists such G_1, G_2, then $G_1 T = I + K$. Therefore $Ker(T) \subset Ker(G_1 T) = Ker(I + K)$ which we claim is finite-dimensional. Indeed, if v_i is a bounded sequence in $Ker(I + K)$, then $Kv_i = -v_i$ has a convergent subsequence. But the unit ball is compact in a Banach space if and only if the space is finite-dimensional (Riesz's lemma). Thus $ker(T)$ is finite dimensional. Likewise, $Coker(T) = Im(T)^{\perp}$ is finite dimensional from $TG_2 = I + \tilde{K}$ and the fact that if K is compact, so is K^*. Thus T is Fredholm.
(5) If F is Fredholm, there exists a G so that $FG = I + K_1$ and $GF = I + K_2$. Now if F were invertible, then $(F + p)^{-1} = F^{-1}(1 + F^{-1}p)^{-1} = F^{-1} \sum (-1)^i (F^{-1}p)^i$ which makes sense if $\|p\|$ is small. Now, define $G_p = G(1 + Gp)^{-1}$ for small p. Now $(F + p)G_p = FG(I + Gp)^{-1} + pG(I + Gp)^{-1} = (I + Gp)^{-1} + K_1(I + Gp)^{-1} + pG(I + Gp)^{-1} = H_p + K$ where $H = (I + Gp)^{-1} + pG(I + Gp)^{-1}$. Clearly when p is small, then H_p is invertible.
Thus $(F + p)G_p = (I + KH_p^{-1})H_p$. Now define $\tilde{G}_p = G_p H_p^{-1}$. So $(F + p)\tilde{G}_p = I + compact$. Likewise we can find another $\tilde{G'}_p$ which is an approximate left inverse for small p. Thus $F + p$ is Fredholm for all small p if F is so.

(6) If we prove that $Ind(F+p) = Ind(F)$ for all small p, we will be done because I is connected. First we prove that for small p, there is a linear transformation $A_p : Ker(T) \to Coker(T)$ so that $Ker(T+p) = Ker(A_p)$ and $Coker(T+p) = Coker(A_p)$. For operators between finite-dimensional spaces, the index equals the difference in dimensions and is hence a constant. Indeed, writing $T : Ker(T)^{\perp} \oplus Ker(T) \to Im(T) \oplus Coker(T)$ as $T = \begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}$ where T' is an isomorphism. Write $p = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Take $A_p = -c(T'+a)^{-1}b + d$. It can be verified that A_p does the job.

(7) If $G_1 T = I + K_1$ and $TG_2 = I + K_2$, then $G_1(T+K) = I + K_1 + G_1 K = I + \textit{compact}$ and likewise. Thus $T+K$ is Fredholm. Now $T + sK$ has locally constant index where $s \in [0,1]$. Hence $Ind(T+K) = Ind(T)$.

□