NOTES FOR 18 JAN (THURSDAY)

1. Recap

(1) Constructed a parametrix for $Lu = f$ when L is elliptic.
(2) Defined and proved a theorem characterising Fredholm operators between Hilbert spaces.

2. Constant-coefficient elliptic operators on the torus

We define the formal adjoint L^*_{form} of L as follows.

Definition 2.1. If $Lu = \sum_{\alpha,p} [A]_{p,\alpha} D^\alpha u$, then define the formal adjoint $L^*_{form} v = \sum_{\alpha,p} [A^*]_{p,\alpha} (-1)^{|\alpha|} D^\alpha v$. It satisfies $\langle Lu, v\rangle_{L^2(S^1 \times S^1 \ldots)} = \langle u, L^*_{form} v\rangle_{L^2(S^1\times S^1 \ldots}$ whenever u, v are smooth functions.

We have the following easy lemma.

Lemma 2.2. *If L is elliptic, then so is L^*_{form}.*

Using the above theorems and some more work we conclude the following.

Theorem 2.3. *If L is elliptic, then*

(1) *$Im(L) \subset H^s$ is closed, and $ker(L) \subset H^{s+l}$ and $coker(L) = \frac{H^s}{Im(L)}$ are finite-dimensional subspaces. (Fredholm's alternative.)*
(2) *$Ker(L)$ consists of smooth functions.*
(3) *Suppose $L : H^l \to L^2$. Then $Coker(L) \simeq Ker(L^* : L^2 \to (H^l)^*)$ consists of smooth functions and $Coker(L) = Ker(L^*_{form})$.*
(4) *If f is in H^s and $u \in L^2$ is a distributional solution of $Lu = f$, then u is in H^{s+l}. (Elliptic regularity.)*

Proof.
(1) By the above theorems, since there is a parametrix for elliptic operators, $L : H^{s+l} \to H^s$ is Fredholm. Hence its kernel and cokernel are finite dimensional and its range is closed.
(2) This follows from the last result in this lemma.
(3) If $u \in (L^2)^* \cap ker(L^*)$. Then $L^*u(v) = u(Lv) = \langle u, Lv\rangle_{L^2} = 0$ for all $v \in H^l$. Thus, choosing v to be a smooth function, we see that u is a distributional solution to $L^*_{form} u = 0$. Since the formal adjoint is also elliptic, by the previous part, its kernel consists of smooth functions.
(4) Suppose $\phi : S^1 \times S^1 \ldots$ is any smooth function. Since $u \in L^2$ is supposedly a distributional solution (by the way u need not be in L^2 for this to be true, it need be only a distribution), $\langle L^*_{form}\phi, u\rangle_{L^2} = \langle \phi, f\rangle_{L^2}$. This means that (by the Parseval-Plancherel theorem), $\sum_{\vec{k}} \hat{\phi}^T \overline{\widehat{L_{form}}} \hat{u} = \hat{\phi}^T \bar{\hat{f}}$ for all ϕ. Now choose ϕ to have Fourier series such that $\hat{\phi}(k) = 1$ if and only if $\vec{k} = \vec{a}$ and 0 otherwise. Then $\widehat{L_{form}}(\vec{a})\hat{u}(\vec{a}) = \hat{f}(\vec{a})\ \forall\ \vec{a}$. This implies that $\hat{u} = \hat{u}_{app}$ for $N = 0$. Hence, by the previous results, $u \in H^{s+l}$.

□

Remark 2.4. The above implies that elliptic operators with constant coefficients on the torus are Fredholm operators between Sobolev spaces. So their index is constant under small (arbitrary) perturbations and under compact perturbations. This index turns out to be given by an integral over the torus of some differential form (whose De Rham cohomology class depends only on the principal symbol of L). This is a special case of the Atiyah-Singer index theorem which deals with general elliptic operators on general manifolds.

3. Riemannian manifolds and metrics on vector bundles

In order to define $\Delta u = f$ on a manifold, unfortunately, we cannot do this locally by choosing coordinates and saying $\sum_i \frac{\partial^2}{\partial (x^i)^2} u = f$ because if we change coordinates, then the PDE will not be the same. So how can hope to even set up the Poisson PDE on a manifold ?

Another way of looking at the Laplacian is $\Delta = \nabla . \nabla$. So if we can define a dot product on every tangent space, and define the ∇ operation, then we can define the Laplacian. Why would we care about defining the Laplacian ? Among other things, it gives insight into the De Rham cohomology of the manifold.

Recall that a smooth vector bundle V over a smooth manifold M is a "smoothly varying collection of vector spaces parametrised by M", i.e., locally, $V \simeq U \times \mathbb{R}^r$ (where instead of $\mathbb{R}$, we can also have $\mathbb{C}$ - such a beast is a complex vector bundle) via a trivialisation, i.e., a collection of smooth sections $e_1, \ldots, e_r : U \subset M \to V$ such that $e_1(p), \ldots, e_r(p)$ form a basis for V_p at all $p \in U$. Equivalently, a vector bundle is simply a collection $(U_\alpha, g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r))$ satisfying $g_{\alpha\alpha} = Id, g_{\alpha\beta} = g_{\beta\alpha}^{-1}, g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = Id$. Fundamental examples of vector bundles are the tangent bundle TM, the cotangent bundle T^*M, and the bundles of differential forms $\Omega^k(M)$. These can be defined using transition functions as in the last semester. Just as before, we will be using the Einstein summation convention. Repeated indices are summed over.

A metric g on a vector bundle V over M is a smooth section of $V^* \otimes V^*$ such that on each fibre it is symmetric and positive-definite. In other words, suppose e_i is a trivialisation of V over U and e^{i*} the dual trivialisation of V^* over U, then $g(p) = g_{ij}(p) e^{i*} \otimes e^{j*}$ where $g_{ij} : U \subset M \to GL(r, \mathbb{R})$ is a smooth matrix-valued function to symmetric positive-definite matrices. So a metric is simply a smoothly varying collection of dot products, one for each fibre. Recall that we proved (in the last semester)

Theorem 3.1. *Every rank-r real vector bundle V over a manifold M admits a smooth metric g.*

In the special case when $V = TM$, the metric is called a Riemannian metric on M. If (x, U) is a coordinate chart, then $g(x) = g_{ij}(x) dx^i \otimes dx^j$. By symmetry, $g_{ij} = g_{ji}$. Moreover, g is a positive definite matrix. If one changes coordinates to y^μ then $g_{\mu\nu} = g_{ij} \frac{\partial x^i}{\partial y^\mu} \frac{\partial x^j}{\partial y^\nu}$. Given a metric g on TM, we get one on T^*M given by $g^* = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Now $g_{ik} g^{kj} = \delta_i^j$.

If M is oriented, supposing (x, U) is an oriented coordinate chart, then $vol = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2 \ldots dx^m$ is a well-defined top form. Indeed, if we changes coordinates, it transforms correctly as seen in the linear algebra above. This is called the “volume” form of the metric.

Here are examples :

(1) Euclidean space $\mathbb{R}^n, g_{Euc} = \sum dx^i \otimes dx^i$. This is the usual metric. The length of a tangent vector v is $\sum (v^i)^2$.
(2) If we take the same Euclidean space $\mathbb{R}^2$ and use polar coordinates, $x = r\cos(\theta)$, $y = r\sin(\theta)$, then $dx = dr\cos(\theta) - r\sin(\theta) d\theta$, $dy = dr\sin(\theta) + r\cos(\theta) d\theta$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta$.
(3) The circle $S^1 : g = d\theta \otimes d\theta$.

(4) If M, g_M, N, g_N are two Riemannian manifolds, then $M \times N, g_M \times g_N$ given by $g_M \times g_N(v_M \oplus v_N, w_M \oplus w_N) = g_M(v_M, w_M) + g_N(v_N, w_N)$. This gives a metric on the n-torus using the circle metric.
(5) The Hyperbolic metric $\mathbb{H}^m, g_{Hyp} : g_{Hyp} = \frac{\sum dx^i \otimes dx^i}{(x^m)^2}$.

Recall the definition of an induced metric

Definition 3.2. If g is a metric on M and $S \subset M$ is an embedded submanifold, then g induces a metric $g|_S$ on S given by $g_p|_S(v_S, w_S) = g_p(i_* v_S, i_* w_S)$.

(1) $S^2 \subset \mathbb{R}^3$. First write the metric in $\mathbb{R}^3$ in spherical coordinates $z = r\cos(\theta)$, $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$. Thus, $g_{Euc} = dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2(\theta) d\phi \otimes d\phi$. Now when we restrict to the unit sphere, the tangent vectors do not include $\frac{\partial}{\partial r}$. Thus, $g_{Sphere} = d\theta \otimes d\theta + \sin^2(\theta) d\phi \otimes d\phi$
(2) Suppose $z = f(x, y)$ is the graph of a function, then $g_{Induced} = dx \otimes dx + dy \otimes dy + (\frac{\partial f}{\partial x})^2 dx \otimes dx + (\frac{\partial f}{\partial y})^2 dy \otimes dy + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(dx \otimes dy + dy \otimes dx)$.

Now we write down the volume forms of most of the above examples :

(1) $vol_{Euc} = dx^1 \wedge dx^2 \wedge \ldots dx^n$.
(2) In polar coordinates in $\mathbb{R}^2$, $vol_{Euc} = \sqrt{\det(g)} dr \wedge d\theta = r dr \wedge d\theta$.
(3) For the circle, $vol = d\theta$.

Suppose $\gamma : [0, 1] \to M$ is a smooth path. Then, define its length as $L(\gamma) = \int_0^1 \sqrt{g(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})} dt$. Since the length has a square root, we consider the Energy $E(\gamma) = \int_0^1 g(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}) dt$. a piecewise C^1 curve satisfying the following equation is called a geodesic. Every critical point of the energy is a geodesic.

$$\begin{aligned} &\frac{d^2\gamma^r}{dt^2} + \Gamma^r_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0 \\ &\Gamma^r_{ij} = g^{rl}\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) \end{aligned} \tag{3.1}$$

We proved that every geodesic is actually smooth. We also proved that every geodesic can be parametrised by its arc-length and that arc-length parametrised geodesics are precisely the critical points of the length functional. We also agreed that $d(p, q) = \inf L(p, q)$ over all the piecewise C^1 paths joining p and q is a metric and that the topology induced by it is the same as the original topology of the manifold. This is where we stopped in the last semester.

Now we note that geodesics exist locally, and that if γ is a geodesic, then so is $\gamma(ct)$. In fact, we have the following result.

Theorem 3.3. *Let $p \in M$. Then there is a neighbourhood U_o of p and a number $\epsilon_p > 0$ such that for every $q \in U$ and every tangent vector $v \in T_qM$ with $\|v\| < \epsilon_p$ there is a unique geodesic $\gamma_v : (-2, 2) \to M$ satisfying $\gamma_v(0) = q$, $\frac{d\gamma_v}{dt}(0) = v$.*

Proof. The fundamental existence and uniqueness theorem for ODE states that for a system of ODE of the form $y^{'} = F(y, t)$, there is a solution for a short period of time, wherein the period depends on the C^1 norm of $F(y, t)$ in a neighbourhood of the initial condition $y(0)$. This is easily seen to imply that there is a neighbourhood $p \in U_p$ and $\epsilon_{1p}, \epsilon_{2p} > 0$ so that for $q\ inU$ and $v \in T_qM$ with $\|v\| < \epsilon_1$, there is a unique smooth geodesic $\gamma_v : (-2\epsilon_2, 2\epsilon_2) \to M$ with the intial conditions. Choose $\epsilon < \epsilon_1\epsilon_2$. Then if $\|v\| < \epsilon$ and $t < 2$, then we can define $\gamma_v(t) = \gamma_{v/\epsilon_2}(\epsilon_2 t)$. $\square$

If $v \in T_qM$ is a vector for which there is a geodesic, $\gamma : [0,1] \to M$ satisfying $\gamma(0) = q$ and $\gamma^{'}(0) = v$ then we define $\exp_q(v) = \gamma_v(1)$. The geodesic itself can be described as $\gamma(t) = \exp_q(tv)$ (by the uniqueness theorem for ODE). By the smooth dependence on parameters of an ODE, $\exp_q(v)$ depends smoothly on q and on v and defines a smooth map $\exp_q : T_qM \to M$.