NOTES FOR 1 FEB (THURSDAY)

1. Recap

(1) Proved the Hopf-Rinow theorem and gave examples of complete manifolds.
(2) Made a few observations regarding the Christoffel symbols, the Riemann curvature tensor, and why we need a notion of parallel transport.

2. Connections and curvature

Here are a bunch of observations / questions :

(1) In $\mathbb{R}^n$, you have the idea of a "constant" vector field. (Indeed, this is one way to prove that $\mathbb{R}^n$ is parallelizable, i.e., it has trivial tangent bundle.) So you need to able to find the directional derivative $\nabla_V X$ of any vector field along a direction V. Note that if we manage to define this concept, then $\nabla_{\gamma'(t)} X(\gamma(t)) = 0$ amounts to parallel transporting the vector field along γ.
(2) Suppose $(S, g_S) \subset (\mathbb{R}^n, Euc)$ is a submanifold with the induced metric. (Actually every Riemannian manifold is of this form by the Nash embedding theorem.) Suppose X is a tangent vector field along S. Suppose that $N_1, N_2 \ldots, N_k$ are local linearly independent unit normal vector fields on $U \subset S$ (where $k = n - dim(S)$). Assume that V is a tangent vector on S at p. How can we define the directional derivative $\nabla_V X(p)$? Clearly, the usual Euclidean directional derivative $D_V X = \frac{\partial \vec{X}}{\partial x^i} V^i$ is not the right one because it measures how fast X is changing perpendicular to S as well. So we need to project this back to S.

   In other words, the "correct" way to define a directional derivative is $\nabla_V X = D_V X - \sum_{i=1}^{k} \langle D_V X, N_i \rangle_{Euc} N_i$. Now note that $\langle D_V X, N_i \rangle_{Euc} = D_V \langle X, N_i \rangle_{Euc} - \langle X, D_V N_i \rangle_{Euc} = -\langle X, D_V N_i \rangle_{Euc}$. In other words,

$$\nabla_V X = D_V X - \sum \langle X, D_V N_i \rangle_{Euc} N_i = D_V X + a\ term\ linear\ in\ X. \tag{2.1}$$

   It turns out (miraculously) that the linear term is related to the Christoffel symbols and the Riemann curvature tensor of g_S that we defined before. This way of defining a directional derivative is called the Levi-Civita connection. In general, a "directional derivative" on a vector bundle is called a "connection".
(3) The above definitions of directional derivative are important even for a general vector bundle. For example if we want to prove that there is a nowhere vanishing section of a certain vector bundle, ideally, we would want to take a "constant" section. But to even define that, we need to know the notion of a directional derivative.
(4) The notion of "curvature" seems to depend on one derivative of the Christoffel symbol (or alternatively, two derivatives of the metric).

The above mentioned observations force us to define a connection $\nabla_W s$ on vector bundles. It is suppose to represent how fast a section s is changing along the tangent vector W. In fact, if W is a vector field, then $\nabla_W s$ better be a section of the vector bundle itself. So, we have

Definition 2.1. Suppose V is a smooth rank-r real vector bundle (a similar definition holds for complex vector bundles) over a smooth manifold M. Suppose $\Gamma(V)$ is the (infinite-dimensional) vector space of smooth sections of V over M. Suppose X is a vector field on M. Then a connection (sometimes called an affine connection) ∇ on V is a map $\nabla_X : \Gamma(V) \to \Gamma(V)$ satisfying the following properties.

(1) *Tensoriality in X* : If s is a smooth section of V, X_1, X_2 are two vector fields, and f_1, f_2 are two smooth functions, then $\nabla_{f_1X_1+f_2X_2}(s) = f_1\nabla_{X_1}s + f_2\nabla_{X_2}s$. In other words, the value of $\nabla_X s$ at p depends only on the value of X at p but not on the derivatives of X.
(2) *Linearity in s* : If s_1, s_2 are two sections and c_1, c_2 are two real numbers, then $\nabla_X(c_1s_1 + c_2s_2) = c_1\nabla_X s_1 + c_2\nabla_X s_2$.
(3) *Leibniz rule* : If f is a smooth function and s is a section, $\nabla_X(fs) = f\nabla_X s + df(X)s = f\nabla_X + X(f)s$.

The first assumption (tensoriality in X) can be stated in another nice way : Suppose we fix s. Then the map $(X, \alpha) \to \alpha(\nabla_X s)$ is a map from $Vect\ fields \times \Gamma(V^*) \to C^\infty\ functions$ which is multilinear (over functions). Therefore, by a theorem we proved the last semester, there exists a smooth section $T_s \in \Gamma(T^*M \otimes V^{**} \simeq V)$ such that $T_s(X, \alpha) = \alpha(\nabla_X s)$.

Thus ∇ can be thought of as a map $\Gamma(V) \to \Gamma(V \otimes T^*M)$ given by $s \to \nabla s$. The space $\Gamma(V \otimes T^*M)$ is commonly called "vector-valued 1-forms". Moreover, in this framework, a connection satisfies $\nabla(fs) = df \otimes s + f\nabla s$.

Locally, suppose $e_1, \ldots, e_r$ is a frame (i.e. a collection of smooth local sections such that every point, they form a basis for the fibre) giving a local trivialisation of V. Then every smooth section is of the form, $s = s^\mu e_\mu$ where s^μ are smooth functions. Therefore,

$$\nabla(s^\mu e_\mu) = ds^\mu \otimes e_\mu + s^\mu \nabla e_\mu = ds^\mu \otimes e_\mu + s^\mu A^\nu_{\ \mu} \otimes e_\nu = (ds^\mu + A^\mu_{\ \nu} s^\nu) \otimes e_\mu \tag{2.2}$$

where $A^\mu_{\ \nu}$ is an $r \times r$ matrix consisting of 1-forms. Note that $\nabla_X s = X(s^\mu)e_\mu + A_{\ \nu}(X)^\mu s^\nu e_\mu$. Suppose we change our trivialisation to $\tilde{e}_1, \tilde{e}_2, \ldots$. Then of course the matrix of 1-forms A will change to $\tilde{A}$. Let us calculate this change. Suppose $\tilde{e}_\mu = g^\nu_{\ \mu} e_\nu$, i.e., $\tilde{e} = eg$ where g is an invertible smooth matrix-valued function. Then since $s = \tilde{s}^\mu \tilde{e}_\mu = s^\nu e_\nu$, we see that $\tilde{e}\tilde{\vec{s}} = eg\tilde{\vec{s}} = e\vec{s}$. Hence $\tilde{\vec{s}} = g^{-1}\vec{s}$. Since $\nabla_X s$ is a section, $\tilde{\vec{\nabla}}_X s = g^{-1}\vec{\nabla}_X s$, i.e., $\tilde{\vec{\nabla}} s = g^{-1}\vec{\nabla} s$. Hence,

$$d\tilde{\vec{s}} + \tilde{A}\tilde{\vec{s}} = g^{-1}(d\vec{s} + A\vec{s}) \Rightarrow d(g^{-1}\vec{s}) + \tilde{A}g^{-1}\vec{s} = g^{-1}(d\vec{s} + A\vec{s})$$
$$-g^{-1}dgg^{-1}\vec{s} + \tilde{A}g^{-1}\vec{s} = g^{-1}A\vec{s} \Rightarrow \tilde{A} = g^{-1}Ag + g^{-1}dg \tag{2.3}$$

In more familiar terms, rewriting $\tilde{\vec{s}} = g\vec{s}$ where g are the transition functions (i.e., replacing g^{-1} by g), we see that $\tilde{A} = gAg^{-1} - dgg^{-1}$.

So A does not change like a tensor. However, the cool thing is that, suppose ∇_1 is one connection. Then, if ∇_2 is any other connection, $(\nabla_2 - \nabla_1)(fs) = f(\nabla_2 - \nabla_1)s$. In other words, the difference of any two connections is an Endomorphism of the vector bundle. Locally, $\tilde{A}_2 - \tilde{A}_1 = g(A_2 - A_1)g^{-1}$. In other words, $A_2 - A_1$ is a section of $End(V) \otimes T^*M$. So the space of connections is an affine space (a vector space without a preferred choice of an origin).