NOTES FOR 1 MAR (THURSDAY)

1. Recap

(1) Defined sectional curvature (determines the Riemann tensor completely) and the notions of positive/negative/constant curvature.
(2) Wrote examples of manifolds with curvatures. Defined Space forms.
(3) Stated the Killing-Hopf, Cartan-Hadamard, Preissman, and Synge theorems.
(4) Defined divergence and gradient.

2. Divergence, Stokes' theorem, and Laplacians

Theorem 2.1. $\int_M div(X)vol_g = \int_{\partial M} i_X vol_g$ *where* $i_X\omega(Y_1, Y_2, \ldots) = \omega(X, Y_1, Y_2, \ldots)$. *If* $\vec{N}$ *is a unit outward pointing normal vector field on the boundary, then* $i_X vol_g = g(X, \vec{N})dvol_{g|_{\partial M}}$.

Proof. Choose oriented normal coordinates x^i for g at $p \in M$. Now

$$div(X)(p)vol_g(p) = \sum_i \frac{\partial X^i}{\partial x^i}(p)dx^1 \wedge dx^2 \ldots dx^m(p) = d(\sum_i X^i(-1)^{i-1}dx^1 \wedge \ldots dx^{i-1}\hat{dx^i} \wedge \ldots)(p)$$
$$= d(i_X vol)(p) \tag{2.1}$$

Since the above equation is an equation of globally defined forms at p, it is independent of coordinates chosen. Thus $div(X)vol_g = d(i_X vol)$. By the usual Stokes' theorem, $\int_M div(X)vol_g = \int_{\partial M} i_X vol$. Now if $X = g(X, \vec{N})\vec{N} + Y$, then Y is tangent to the boundary. Choose oriented normal coordinates such that $x^1 = 0$ corresponds to the boundary (hence $\vec{N}(p) = \frac{\partial}{\partial x^1}$ and Y is a linear combination of ∂_i where $i \geq 2$)Then $i_X vol(p)|_{x^1=0} = g(X, \vec{N})(p)i_{\vec{N}(p)}dx^1 \wedge dx^2 \ldots (p) + i_{Y(p)}dx^1 \wedge dx^2 \ldots (p) = g(X, \vec{N})vol_{g|_{\partial M}}(p)$. As before, this equation holds globally. ∎

In particular, if M has no boundary, then $\int_M div(X) = 0$. Now define

Definition 2.2. The Laplacian Δu where u is a function on M is a function $\Delta u = div(\nabla u) = \frac{\partial}{\partial x^i}\left(g^{ij}\frac{\partial u}{\partial x^j}\right) + \Gamma^i_{ik}\frac{\partial u}{\partial x^j}g^{jk}$. So in normal coordinates, it is the usual Laplacian at p.

As an example, take the flat metric $g = d\theta^1 \otimes d\theta^1 + d\theta^2 \otimes d\theta^2 + \ldots$ on the torus. Then the Laplacian is easily seen to be the Laplacian we studied earlier. Here is an observation using Stokes :

$$\int_M \Delta u = \int div(grad(u))dvol_g = 0 \tag{2.2}$$

So if $\Delta u = f$, a necessary condition is that $\int f dvol_g = 0$ (just like the torus). If $\Delta u = f$, then observe that for any smooth function v,

$$\int_M v\Delta u vol_g = \int_M v f vol_g \Rightarrow \int_M (div(v\nabla u) - \nabla v.\nabla u)vol_g = -\int_M \nabla v.\nabla u vol_g = \int_M u\Delta v vol_g \tag{2.3}$$

So we can define a distributional solution of $\Delta u = f$ as an L^2 function u such that the above holds for all smooth v.

What about the curl of a vector field X ? Firstly, given a vector field X, we can produce its dual

1-form $\omega_X(Y) = g(X,Y)$. We can then define $d\omega_X$ as a 2-form. If there is a way to take a 2-form α to an $m-2$ form $*\alpha$, then in 3-dimensions, $*\alpha$ will be a 1-form, whose dual is a vector field. This should be the curl. So we need a notion called the Hodge star $*$ taking k-forms to $m-k$ forms.

Definition 2.3. Given a k-form α on a compact oriented m-dimensional Riemannian manifold (M,g), $*\alpha$ is a $(m-k)$-form such that $\alpha\wedge*\beta = \langle\alpha,\beta\rangle_g vol_g$. Here the inner product on forms is defined as follows : Suppose at p, normal coordinates are chosen, i.e., $g_{ij}(p) = \delta_{ij}$, then $dx^{i_1}(p)\wedge dx^{i_2}\ldots\wedge dx^{i_k}(p)$ form an orthonormal basis at p for k-forms. Note that $vol(p) = dx^1(p)\wedge dx^2(p)\ldots dx^m(p)$.

Does such an operator $*:\Gamma(\Omega^k(M))\to\Gamma(\Omega^{m-k}(M))$ exist ? Is it linear ? Yes to both. Suppose $\omega_1,\omega_2,\ldots,\omega_m$ form an orthonormal frame on an open set U, i.e., $\omega_1(p),\omega_2(p),\ldots,\omega_m(p)$ form an orthonormal basis of T_p^*M for all $p\in U$. Then, $*(\omega_{i_1}\wedge\omega_{i_2}\ldots\omega_{i_k}) = (-1)^{sgn(I)}\omega_{i_{k+1}}\wedge\omega_{i_{k+2}}\ldots\wedge\omega_{i_m}$ where $sgn(I)$ is the sign of the permutation taking $(1,2,\ldots,m)$ to $(i_1,i_2,\ldots,i_m)$. Then extend $*$ linearly to all forms. We will see why it is well-defined later on. Here are some examples :

(1) Suppose $(M,g) = \mathbb{R}^2, g_{Euc}$ oriented in the usual way, then $*1 = dx\wedge dy$. Also, $*dx = dy$ and $*dy = -dx$. Finally, $*(dx\wedge dy) = 1$.
(2) If $M = \mathbb{R}^3$ (with the Euclidean metric) oriented in the usual way, then $*1 = dx\wedge dy\wedge dz$, $*dx = dy\wedge dz$, $*dy = dz\wedge dx$, $*dz = dx\wedge dy$.
If $\vec{v} = (v_1,v_2,v_3)$ and $\vec{w} = (w_1,w_2,w_3)$, form the dual 1-forms $v = v_1dx + v_2dy + v_3dz$ and likewise for w. Then $v\wedge w$ is a 2-form given by $v\wedge w = (v_1w_2 - v_2w_1)dx\wedge dy + \ldots$. The Hodge star acting on this gives a 1-form $*(v\wedge w) = (v_1w_2 - v_2w_1)dz + \ldots$ whose dual is $(v_2w_3 - v_3w_2, v_3w_1 - w_3v_1, v_1w_2 - w_1v_2)$ which are the components of $\vec{v}\times\vec{w}$. Since the cross product depends on the choice of orientation, it is called a "pseudovector".

This $*$ operator (the so-called Hodge star) has the following properties :

(1) Suppose α,β are elements of $\Omega_p^k(M)$ $\alpha\wedge*\beta = \langle\alpha,\beta\rangle_g vol_g = \beta\wedge*\alpha$, i.e., it does satisfy the definition.
(2) $*$ is well-defined, i.e., it does not depend on the choice of orthonormal basis.
(3) If you change the metric from g to $\tilde{g} = cg$ where $c>0$ is a constant, then $*_{\tilde{g}}\omega = \sqrt{c}^{2k-m} *_g \omega$
(4) If you change the orientation, $*\to -*$.
(5) $**\eta = (-1)^{k(m-k)}\eta$.
(6) $\langle *\alpha, *\eta\rangle = \langle\alpha,\eta\rangle$.

Proof. (1) Suppose we choose the orthonormal frame ω_i. Suppose $\beta = \beta_I\omega^{i_1}\wedge\omega_{i_2}\ldots$ where the summation is over increasing indices $i_1 < i_2 < \ldots$, we see that $*\beta = \beta_I(-1)^{sgn(I)}\omega^{i_{k+1}}\wedge\omega^{i_{k+2}}\ldots$. Thus,

$$\alpha\wedge*\beta = \alpha_J\beta_I(-1)^{sgn(I)}\omega^{j_1}\wedge\omega^{j_2}\ldots\omega^{j_k}\wedge\omega^{i_{k+1}}\wedge\ldots$$
$$= \alpha_I\beta_I(-1)^{sgn(I)}(-1)^{sgn(I)}\omega_1\wedge\omega_2\ldots = \alpha_I\beta_I vol_g = \langle\alpha,\beta\rangle vol_g \tag{2.4}$$

Note that this property does not depend on how we defined $*$ (i.e., we did not use the fact that $*$ is well-defined)

(2) The above property $\alpha\wedge*\beta = \langle\alpha,\beta\rangle vol_g$ *defines* $*$ uniquely because, if $*_1, *_2$ satisfy this property, then $\alpha\wedge(*_1 - *_2)\beta = 0$ for all α,β. However, $(a,b)\to a\wedge b$ is a non-degenerate pairing (Why? because $(a, *_1a)\to a\wedge*_1a = |a|^2vol_g \geq 0$). Hence $*_1\beta = *_2\beta\ \forall\ \beta$.
(3) Suppose $\omega_1,\ldots,\omega_m$ is an orthonormal frame for g, then $\frac{\omega_i}{\sqrt{c}}$ is one for $\tilde{g}$. From this the result follows trivially.
(4) Obvious.

(5)

$$**(\eta) = \eta_I **(\omega^I) = \eta_I * ((-1)^{sgn(I)}\omega^{I^c}) = \eta_I(-1)^{sgn(I)}(-1)^{sgn(I^c)}\omega^I = (-1)^{k(m-k)}\eta \tag{2.5}$$

(6) Suppose η is a k-form and α an $m-k$ form.

$$\begin{aligned}\langle *\alpha, *\eta\rangle vol &= *\alpha \wedge **\eta = (-1)^{k(m-k)} * \alpha \wedge \eta \\ &= (-1)^{k(m-k)}(-1)^{k(m-k)}\eta \wedge *\alpha = \langle \eta, \alpha\rangle vol = \langle \alpha, \eta\rangle vol \end{aligned} \tag{2.6}$$

□

Now we define an operator analogous of the curl $\nabla \times \vec{F}$:

Definition 2.4. Let α be a smooth k-form. Then $d^\dagger\alpha = (-1)^{m(k+1)+1} * d * \alpha$. Thus $d^\dagger\alpha$ is a smooth $k-1$-form depending on the first derivative of α (it is a first order differential operator).