NOTES FOR 20 MAR (TUESDAY)

1. Recap

(1) Proved Sobolev embedding and compactness.
(2) Proved a structure theorem for linear PDOs, defined ellipticity, and stated the regularity theorem. Started proving for weak solutions it assuming it holds for smooth solutions.

2. Elliptic regularity

Theorem 2.1. *If L is uniformly elliptic and f a smooth section of F. Then if $u \in L^2$ satisfies $Lu = f$ in the sense of distributions then u is smooth. Moreover, if $f \in H^s$, then $u \in H^{s+l}$ and $\|u\|_{H^{s+o}} \leq C_s(\|f\|_{H^s} + \|u\|_{L^2})$ where C_s depends only on h_E, h_F, g, ∇_E, an upper bound on $\|a_k\|_{C^{s+o}}$, and on the ellipticity constants.*

We claim that this theorem follows from

Theorem 2.2. *If L is uniformly elliptic, u is a smooth section of E, then $\|u\|_{H^{s+o}} \leq C_s(\|Lu\|_{H^s} + \|u\|_{L^2})$.*

This claim will not be proved (because it requires some work done using distributions). The proof of not just the claim, but of the full theorem is in Folland's book for instance. Instead, the weaker theorem will be proved.

Proof. Writing $u = \sum \rho_\mu u$, we see that if we can prove that $\|\rho_\mu u\|_{H^{s+o}} \leq C_s(\|L(\rho_\mu u)\|_{H^s} + \|\rho_\mu u\|_{L^2})$ we will be done. Indeed (from now onwards all constants depending on s (and on the ellipticity constants and upper bounds on the coefficients) will be denoted by abuse of notation as C_s),

$$\|u\|_{H^{s+l}} \leq C_s \sum_\mu (\|L(\rho_\mu u)\|_{H^s} + \|\rho_\mu u\|_{L^2}) \leq \tilde{C}_s \sum_\mu (\|\rho_\mu L u\|_{H^s} + \|u\|_{L^2}) + C_s \sum_\mu \|[L, \rho_\mu] u\|_{H^s}$$

$$\leq \tilde{C}_s(\|Lu\|_{H^s} + \|u\|_{L^2}) + C_s \|u\|_{H^{s+l-1}} \tag{2.1}$$

Using the interpolation inequality 2.6 we see that $C_s\|u\|_{H^{s+l-1}} \leq \frac{1}{2}\|u\|_{H^{s+l}} + C\|u\|_{L^2}$. Thus we have reduced the problem to proving $\|\rho_\mu u\|_{H^{s+o}} \leq C_s(\|L(\rho_\mu u)\|_{H^s} + \|\rho_\mu u\|_{L^2})$.

Let $p_\mu \in U_\mu$ be a fixed collection of points. Suppose $\tilde{\rho}_\mu$ is a bump function equal to 1 on the support of ρ_μ and having support in U_μ, then if the cover U_μ is chosen to be fine enough so that the coefficients of L do not vary much from their values at p_μ (the size of this cover will of course depend on the ellipticity constants and an upper bound on the derivatives of the coefficients), then $\tilde{L}_\mu = \tilde{\rho}_\mu L + (1 - \tilde{\rho}_\mu) a_o(p_\mu)^I \partial_I$ can be thought of as a uniformly elliptic operator (with bounded ellipticity constants) acting on the torus with variable coefficients. Thus, we have reduced the problem to proving the estimate on a flat torus with the trivial vector bundle (but with variable coefficients).

The rough idea is to cover the torus with lots of open sets such that the operator is not far from a constant coefficient one on those sets, i.e., $L - L(p_\mu)$ is small. Then we know that $\|\rho_\mu u\|_{H^{s+o}} \leq C_s(\|L(p_\mu)\rho_\mu u\|_{H^s} + \|\rho_\mu u\|_{L^2}) \leq C_s(\|L\rho_\mu u\|_{H^s} + \|\rho_\mu u\|_{L^2}) + C_s\|(L - L(p, \mu))\rho_\mu u\|_{H^s}$. Note that a single

C_s works independent of what p_μ are simply because C_s depends only on the ellipticity constants and upper bounds on the coefficients. If the last term is smaller than $\frac{1}{2}\|u\|_{H^{s+o}}$ (for instance), then we are done. But if we make the cover small, we risk making the other factor large. This is the problem.

Firstly, we claim that it is enough to prove the estimate for $s = 0$. Indeed, if this is done, then

$$\begin{aligned} \|\partial_i u\|_{H^o} &\leq C(\|L(\partial_i u)\|_{L^2} + \|\partial_i u\|_{L^2}) \leq C(\partial_i(Lu)\|_{L^2} + \|[L,\partial_i]u\|_{L^2} + \|u\|_{H^o}) \\ &\Rightarrow \|u\|_{H^{o+1}} \leq C(\|Lu\|_{H^1} + \|u|_{L^2}) \end{aligned} \tag{2.2}$$

Inductively we can prove this for a general s.

Suppose we choose a fine enough cover of the torus so that $\|(L - L(p_\mu))u\|_{L^2} \leq \frac{1}{2C_0}\|u\|_{H^0}$. Then of course $\|(\rho_\mu(L - L(p_\mu))u\|_{L^2} \leq \frac{1}{2C_0}\|u\|_{H^o}$ because $\rho_\mu \leq 1$. Fix such a cover and a partition-of-unity (we will not make it any finer than this). Therefore,

$$\begin{aligned} \frac{1}{2}\|u\|_{H^o} &\leq \sum_\mu C_0(\|L\rho_\mu u\|_{L^2} + \|\rho_\mu u\|_{L^2}) + C_0\sum_\mu \|[(L - L(p,\mu)),\rho_\mu]u\|_{H^s} \\ &\leq C_0\sum_\mu(\|L\rho_\mu u\|_{L^2} + \|\rho_\mu u\|_{L^2}) + C_1\|u\|_{H^{o-1}} \end{aligned}$$

$$\leq C_0(\|Lu\|_{L^2} + \|u\|_{L^2}) + C_0\sum_\mu \|[L,\rho_\mu]u\|_{L^2} + C_1\|u\|_{H^{o-1}} \leq C_0(\|Lu\|_{L^2} + \|u\|_{L^2}) + C_2\|u\|_{H^{o-1}} \tag{2.3}$$

If we can prove that $\|u\|_{H^{o-1}} \leq \frac{1}{3C_2}\|u\|_{H^o} + C\|u\|_{L^2}$, we will be done. Indeed, this follows from the following interpolation inequality for Sobolev spaces.

Lemma 2.3. *If $s^{''} < s^{'} < s$, then for any $f \in H^s(S^1 \times S^1 \ldots, \mathbb{R}^r)$, for any $t > 0$,*

$$\|f\|_{s^{'}}^2 \leq \frac{s^{'} - s^{''}}{s - s^{''}} t^{(s-s^{''})/(s^{'}-s^{''})}\|f\|_s^2 + \frac{s - s^{'}}{s - s^{''}} t^{-(s-s^{''})/(s-s^{'})}\|f\|_{s^{''}}^2 \tag{2.4}$$

Proof. Firstly, we notice the following useful fact of life : If $a, b, t > 0$, $0 \leq \lambda \leq 1$, then

$$a^\lambda b^{1-\lambda} \leq \lambda t^{1/\lambda} a + (1-\lambda) t^{-1/(1-\lambda)} b \tag{2.5}$$

This fact follows from the weighted AM-GM inequality which in turn follows from the method of Lagrange multipliers. Using this fact,

$$(1 + |k|^2)^{s^{'}} \leq \frac{s^{'} - s^{''}}{s - s^{''}} t^{(s-s^{''})/(s^{'}-s^{''})}(1 + |k|^2)^s + \frac{s - s^{'}}{s - s^{''}} t^{-(s-s^{''})/(s-s^{'})}(1 + |k|^2)^{s^{''}} \tag{2.6}$$

Using this it is easy to see the desired Sobolev space inequality. ☐

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