NOTES FOR 22 MAR (THURSDAY)

1. Recap

(1) Proved elliptic regularity for L^2 distributional solutions.

2. Elliptic operators-Fredholmness

We shall prove that

Theorem 2.1. *If $L : H^o \to L^2$ is elliptic, then*

(1) *$Im(L) \subset L^2$ is closed, and the kernel and cokernel are finite-dimensional.*
(2) *The kernel consists of smooth functions.*
(3) *The Cokernel $\simeq ker(L^*) : L^2 \to (H^o)^*$ consists of smooth functions and $Coker(L) \simeq ker(L^*_{form})$.*

Proof. (1) This will follow from the construction of parametrices.
Firstly, we have the following lemma :

Lemma 2.2. *If U_μ is a coordinate trivialising open cover of M, ρ_μ^2 is a partition-of-unity subordinate to it, and $K_\mu : H^s(S^1 \times S^1 \ldots, \mathbb{R}^r) \to H^s(S^1 \times \ldots, \mathbb{R}^r)$ are compact operators, then $K : H^s(M, E) \to H^s(M, E)$ given by $K(u) = \displaystyle\sum_\mu \rho_\mu K_\mu(\rho_\mu u)$ is also compact where we secretly extend functions supported on U_μ to $S^1 \times S^1 \ldots$ and conversely, functions on $S^1 \times S^1 \ldots$ having support in the image of U_μ are extended by 0 to the manifold.*

Proof. Indeed, if u_n is bounded sequence in H^s, then $\rho_\mu u_n$ is bounded in $H^s(S^1 \times \ldots)$ and hence there is a convergent subsequence of $K_\mu(\rho_\mu u_n)$ (depending on μ which we will as usual denote by the subscript n shamelessly) converging to u_μ. Since $\rho_\mu u_\mu$ has compact support, it can be extended to all of M and the convergence happens in the Sobolev norm on M. Thus $K(u_n) = \sum_\mu \rho_\mu K_\mu(\rho_\mu u_n) \to \sum_\mu \rho_\mu u_\mu$ in H^s. Hence K is compact. □

Another small observation is that if $TG_1 - I = K_1, G_2 T - I = K_2$, then $T + h$ is Fredholm for all h satisfying $\|h\| < \delta$ where δ depends only on upper bounds for $\|G_1\|, \|G_2\|$. Lastly, going over the construction of the parametrices on the Torus, their norms are bounded above depending only on the ellipticity constants and upper bounds on the coefficients.

First we prove that elliptic operators with variable coefficients (on a trivial bundle) on a flat unit torus are Fredholm. Indeed, cover the torus with fine enough open sets U_μ, take a partition-of-unity ρ_μ^2, and choose points p_μ. We will decide how many such sets we need later on. Let G_μ be the parametrices from $L^2(S^1 \times S^1 \ldots) \to H^s(S^1 \times S^1, \ldots)$ for $L(p_\mu)$. Note that the norms of G_μ depend solely on the ellipticity constants and upper bounds for the coefficients, and therefore are independent of the size of the open cover U_μ (By the way, we fix these unit tori once and for all corresponding to some *fixed* open cover. All finer open covers contain are subordinate to the fixed one. Hence, the equivalence between the norms of functions supported on U_μ is also fixed. That is, we may freely move between $H^s(S^1 \times S^1 \ldots)$ and $H^s(M,E)$.). Now define $G = \sum_\mu \rho_\mu G_\mu \rho_\mu$. Note that $LG(u) : L^2(M,E) \to L^2(M,E)$ as

$$\begin{aligned} LG(u) &= \sum_\mu [L, \rho_\mu] G_\mu \rho_\mu u + \rho_\mu (L - L_\mu) G_\mu (\rho_\mu u) + \rho_\mu^2 u + \rho_\mu K_\mu \rho_\mu u \\ &= u + Compact\ u + \sum_\mu \rho_\mu (L - L_\mu) G_\mu (\rho_\mu u) \end{aligned} \tag{2.1}$$

Note that (by Cauchy-Schwarz)

$$\begin{aligned} \| \sum_\mu \rho_\mu (L - L_\mu) G_\mu (\rho_\mu u) \|_{L^2} &\leq \Big(\sum_\mu \|L - L_\mu\|^2_{H^o(U_\mu) \to L^2(U_\mu)} \|G_\mu\|^2_{L^2(S^1 \times \ldots) \to H^o(S^1 \times \ldots)} \|\rho_\mu u\|^2_{L^2} \Big)^{1/2} \\ &\leq \frac{\|u\|_{L^2}}{2} \end{aligned} \tag{2.2}$$

if the cover is chosen fine enough. Hence $I + \sum_\mu \rho_\mu (L - L_\mu) G_\mu \rho_\mu$ is invertible and thus there exists a $\tilde{G}_1$ so that $L\tilde{G}_1 - I = Compact$. This proves that the cokernel (as a subset of L^2) is finite dimensional.

To find a right parametrix $\tilde{G}_2$, i.e., $\tilde{G}_2 L = I + K_2$, we need distributions because it is easier to do it for $\tilde{G}_2 : H^{-o} \to L^2$ rather than $\tilde{G}_2 : L^2 \to H^o$.

So we develop the necessary theory : The space of distributions of order s $H^{-s}(M,E)$ is defined to be the metric space completion of L^2 under the norm $\|v\|_{H^{-s}} = \|F_v\| = \sup_{u \in H^s} \frac{|(u,v)_{L^2}|}{\|u\|_{H^s}}$. It is not hard to prove using functional analysis that $(H^s)^* \simeq H^{-s}$. Note that if $v \in L^2$, then $\|v\|_{-s} \leq \|v\|_{L^2}$. Also, if $v \in H^{-s}$, then $|v(u)| \leq \|v\|_{-s} \|u\|_s$. It is not hard to see that the isomorphism $(H^s)^* \simeq H^{-s}$ is an isometry and hence H^{-s} is a Hilbert space. Some easy functional analysis also allows one to conclude that given $G \in (H^{-s})^*$, there is a unique $u \in H^s$ such that $G(v) = (u,v)_{L^2}$. As in the case of a torus, we define the derivative of a distribution through "integration-by-parts". The Sobolev inclusion and Rellich compactness still hold.

In fact, we claim that $v \in H^{-s}$ if and only if $\rho_\mu v \in H^{-s}(S^1 \times S^1 \ldots, \mathbb{R}^r)$ (where $\rho_\mu v(u) = v(\rho_\mu u)$) where U_μ is a trivialising coordinate cover and ρ_μ is a partition-of-unity). Moreover, the H^{-s} norm is equivalent to $\sum_\mu \|\rho_\mu v\|_{H^{-s}(S^1 \times S^1 \ldots)}$. This will be part of a HW. [Actually, using the above technique one can prove elliptic regularity estimates in these more general norms (using the corresponding results for the torus).]

Now we return to elliptic operators. Note that L can be extended using distributional derivatives. In fact, if $u \in L^2$ and $v \in H^o$, then $L(u) \in H^{-o}$ and $\langle L(u), v \rangle = (u, L^*_{form} v)_{L^2}$. In other words, it coincides with $L^{*\dagger}_{form} : L^2 \to H^{-o}$. Now G_μ can be easily extended from $H^{-o}(S^1 \times \ldots) \to L^2(S^1 \times \ldots)$ and it is still a parametrix (whose norm is bounded above depending only on the ellipticity constants and upper bounds on the coefficients). The definition $G = \sum \rho_\mu G_\mu \rho_\mu$ by the "HW results" makes sense as a map from $H^{-o}(M,E) \to L^2(M,E)$. We need to then prove that $\| \sum_\mu \rho_\mu G_\mu (L - L_\mu)(\rho_\mu u) \|_{L^2} \leq \frac{1}{2} \|u\|_{L^2}$ to be done. Indeed (by Cauchy-Schwarz),

$$\begin{aligned} \| \sum_\mu \rho_\mu G_\mu (L - L_\mu)(\rho_\mu u) \|_{L^2} &\leq \Big(\|G_\mu\|^2_{H^{-o} \to L^2} \|L - L_\mu\|^2_{L^2 \to H^{-o}} \|\rho_\mu u\|^2_{L^2} \Big)^{1/2} \\ &\leq \frac{1}{2} \|u\|_{L^2} \end{aligned} \tag{2.3}$$

for a fine enough cover. This proves that the kernel of L (as a subset of L^2) is finite dimensional. By the regularity theorem for L^2 distributional solutions, every element of the kernel is actually smooth (and is hence in H^o as well). So $L : H^o \to L^2$ has finite dimensional kernel and cokernel. Thus it is Fredholm (and hence its range is closed). This means that if f is smooth, $Lu = f$ has a smooth solution if and only if f is L^2 orthogonal to the kernel of L^* (which by the next two results corresponds to being L^2 orthogonal to the kernel of L^*_{form}).

(2) This follows from elliptic regularity that we proved earlier.

(3) If $u \in ker(L^*)$, then $L^*u(v) = u(Lv) = \langle u, Lv\rangle_{L^2} = 0 \ \forall \ v \in H^o$. Choosing v to be smooth, u is a distributional solution of $L^*_{form}u = 0$. By elliptic regularity u is smooth (the formal adjoint is also elliptic).

□

3. Elliptic operators - Diagonalisability

Suppose L is elliptic and symmetric of order $2o$ satisfying Garding's coercivity inequality : $(Lu, u)_{L^2} + \lambda(u, u)_{L^2} \geq \delta(u, u)^2_{H^o}$ (for some positive λ) for all smooth functions. Also assume that $C\|u\|^2_{H^o} \geq B[u, u] = (Lu, u)_{L^2} + \lambda(u, u)_{L^2} \geq \delta\|u\|^2_{H^o}$. Note that $B[u, v]$ makes sense when u is smooth and $v \in L^2$. Fix $v \in H^o$. By approximation with smooth functions, $B[, v]$ extends to H^o. It continues to be symmetric. By Riesz representation, $B[u, v] = (Au, v)_{H^o}$ where the unique $A : H^o \to H^o$ is bounded. It can be seen that A is self-adjoint. Indeed, $(Au, v) = B[u, v] = B[v, u] = (Av, u)$. Of course A coincides with L on smooth functions.

Clearly, B still satisfies the above inequalities. Suppose $f \in L^2$. Since B is symmetric, $B[u, v]$ is a new inner product on H^o which is equivalent to the Sobolev norm and hence Riesz representation implies that for every $f \in L^2$, there is a $u \in H^o$ such that $B[u, v] = (f, v) \ \forall \ v \in H^o$. In particular, u is a solution to $Au = f$. Thus, if v is smooth, then $(Au, v) = (u, Av) = (u, Lv + \lambda v) = (f, v)$. Thus, $u \in H^o$ is a distributional solution to $Lu + \lambda u = f$. Hence it is smooth if f is so.