NOTES FOR 23 JAN (MONDAY)

1. Recap

(1) Proved Fredholm's alternative and Elliptic regularity.
(2) Recalled the definitions of vector bundles, metrics, geodesics and some properties. Defined the exponential map.

2. Riemannian manifolds and metrics on vector bundles

This section is largely from Spivak's book. Note that $(\exp_q)_{v*} : T_v(T_qM) \simeq T_qM \to T_{\exp_q(v)}M$ is its pushforward. We claim that

Theorem 2.1. *$(\exp_q)_{0*} = Id$ and hence $\exp_q$ is a local diffeomorphism around $\vec{0}$.*

Proof. Clearly the first statement and the inverse function theorem imply the second. Now if $v \in T_qM$, we need to obtain a curve $c(t) \in T_qM$ such that $c(0) = 0$, and $\frac{d \exp_q(c(t))}{dt}|_{t=0} = v$. Let $c(t) = tv$. Then $\exp_q(c(t)) = \exp_q(tv)$ which is the time-t geodesic starting at q pointing along v at $t = 0$. Thus we are done. ☐

In fact, we can say more.

Theorem 2.2. *For every $p \in M$ there is a neighbourhood W_p and a number $\epsilon_p > 0$ such that*

(1) *Any two points of W are joined by a unique geodesic in M of length $< \epsilon_p$.*
(2) *Let $v(q, q')$ denote the unique vector $v \in T_qM$ such that $\exp_q(v) = q'$. Then $(q, q') \to v(q, q')$ is smooth.*
(3) *For each $q \in W_p$, the map $\exp_q$ maps the open ϵ-ball in T_qM diffeomorphically onto an open set $U_q \subset W$.*

Proof. Let $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ be coordinates on TM near $(p, 0)$ where p corresponds to the origin. A previous theorem states the exponential map is well-defined in a neighbourhood V of $(p, 0)$ in TM. Define $F : V \to M \times M$ as $F(w) = (\pi(w), \exp_{\pi(w)}(w))$. This is smooth.

If we prove that $F_{(p,0)*}$ is invertible, then by the inverse function theorem, F is a local diffeomorphism. Now choose W to be a smaller neighbourhood of p such that F^{-1} exists and is smooth on $W \times W$.

To prove the invertibility of the derivative, we use coordinates.

$$\frac{\partial F}{\partial x^i}|_{(0,0)} = (\delta_{ij}, \frac{\partial \exp_x(v)}{\partial x^i}|_{(0,0)}) = (\delta_{ij}, \delta_{ij})$$
$$\frac{\partial F}{\partial v^i}|_{(0,0)} = (0, \frac{\partial \exp_x(v)}{\partial x^i}|_{(0,0)}) = (0, Id) \tag{2.1}$$

This is obviously invertible. ☐

Now we make a definition of a useful coordinate system.

Definition 2.3. Given $q \in M$, the coordinate system defined by $\exp_q : U \subset T_qM \to M$ is called a geodesic normal coordinate system at q (after choosing coordinates on U that is).

This set of coordinates is extremely useful. In fact,

Theorem 2.4. *There is a geodesic normal coordinate system v at p, $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial v^k}(p) = 0$.*

Proof. Choose coordinates x^μ so that $g_{\mu\nu}(p) = \delta_{\mu\nu}$. (This can be easily accomplished by taking any coordinate system and rotating it so as to diagonalise g.) Let v^i be coordinates in T_pM. Now exp is a local diffeomorphism. So $x^\mu(v^j) = x^\mu \circ \exp(v^j)$ is a change of coordinates in a small neighbourhood.

Note that since $\exp_{0*} = Id$, $\frac{\partial x^\mu}{\partial v^j}|_{v=0} = \delta^\mu_j$. Now $\tilde{g}_{ij} = g_{\mu\nu}\frac{\partial x^\mu}{\partial v^i}\frac{\partial x^\nu}{\partial v^j}$. So it is easy to see that $\tilde{g}_{ij}(0) = \delta_{ij}$. Since the geodesics through p are linear in this coordinate system, we see that the Christoffel symbols $\tilde{\Gamma}^r_{ij}(0) = 0$. It is easy to see that if the Christoffel symbols are 0, then so are all first partial derivatives of the metric. $\square$

More generally, any coordinate system in which the metric at p is standard upto first order is called a normal coordinate system at p.

Actually, we can prove the existence of normal coordinates in much simpler manner even without reference to geodesics.

Theorem 2.5. *There is a normal coordinate system y at p.*

Proof. Choose any coordinate system at x at p such that $x = 0$ is p. Using a linear map, we may diagonalise g at p. So without loss of generality, $\tilde{g}_{\mu\nu} = \delta_{\mu\nu} + a_{\mu\nu\alpha}x^\alpha + O(x^2)$. (Note that $a_{\mu\nu\alpha} = a_{\nu\mu\alpha}$.) Change the coordinates to y such that $x(y)^i = y^i + b^i_{jk}y^jy^k$ where $b^i_{jk} = b^i_{kj}$. Now

$$\begin{aligned} g_{ij} = \tilde{g}_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} &= (\delta_{\mu\nu} + a_{\mu\nu\alpha}y^\alpha + O(y^2))(\delta^\mu_i + b^\mu_{ik}y^k)(\delta^\nu_j + b^\nu_{jk}y^k) \\ &= \delta_{ij} + a_{ijk}y^k + (b_{ijk} + b_{jik})y^k + O(y^2) \end{aligned} \tag{2.2}$$

So we just need to choose b so that $a_{ijk} = -b_{ijk} - b_{jik}\ \forall\ k$. So take $b = -\frac{a}{2}$. $\square$

It is natural to ask if there is a geodesic normal coordinate system to the second order. Shockingly enough, there isn't (in general). In fact,

Theorem 2.6. *There exists a $(0,4)$ tensor (called the Riemann curvature tensor of g) which is locally $R_{\mu\nu\alpha\beta}$ such that in geodesic normal coordinates,*

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}(0)x^\mu x^\nu + O(x^3) \tag{2.3}$$

where in these coordinates, $R_{ijkl}(0) = \frac{1}{2}\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l}(0) + \frac{1}{2}\frac{\partial^2 g_{il}}{\partial x^j \partial x^k}(0) - \frac{1}{2}\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k}(0) - \frac{1}{2}\frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}(0)$. In fact, all the other terms in the Taylor expansion depend only on R and its derivatives. So there is a change of coordinates such that g is Euclidean everywhere, then, since the Euclidean coordinates are geodesically normal, the Riemann curvature tensor is identically 0.

So one can prove that one cannot draw a map of any part of Bangalore on a piece of paper such that distances are to scale, by calculating the curvature of the sphere with the metric induced from the Euclidean space. It turns out to be a non-zero tensor. We will return to curvature later on in a different way. This theorem is to show you that the notion of curvature is "forced" upon us. (It is not an artificial definition.)

Now we need to know that geodesics are locally length minimising. For this we need to prove the folllowing lemma due to Gauss.

Lemma 2.7. *In U_q, the geodesics through q are perpendicular to the hypersurfaces $\{\exp_q(v) : \|v\| = c < \epsilon\}$*

Proof. Let $v : \mathbb{R} \to T_qM$ be a smooth curve with $\|v(u)\| = k < \epsilon \ \forall\ u$. Define $\beta(u,t) = \exp_q(tv(u))$. Now β is a variation of the geodesic $\gamma(t) = \exp_q(tv(0))$ (but the endpoints are not fixed). By the first variation formula,

$$\frac{dE(\beta)}{du}|u = 0 = -\langle \frac{\partial \beta}{\partial u}(0,1), \frac{d\gamma}{dt}(1)\rangle - \langle \frac{\partial \beta}{\partial u}(0,0), \frac{d\gamma}{dt}(0)\rangle = -\langle \frac{\partial \beta}{\partial u}(0,1), \frac{d\gamma}{dt}(1)\rangle \tag{2.4}$$

But each curve $\beta(u)$ has energy $E(\beta(u)) = \int_0^1 \frac{\partial \beta(u)(t)}{\partial t}\|^2 dt = \int_0^1 k^2 dt = k^2$. Hence $0 = \frac{\partial E}{\partial u} = -\langle \frac{\partial \beta}{\partial u}(0,1), \frac{d\gamma}{dt}(1)\rangle$. This proves the result. $\square$

As a corollary we see that

Corollary 2.8. *Let $c : [a,b] \to U_q - \{q\}$ be a piecewise smooth curve $c(t) = \exp_q(u(t)v(t))$ for $0 < u(t) < \epsilon$ and $\|v(t)\| = 1$. Then $L_a^b c \geq |u(b) - u(a)|$ with equality holding if and only if u is monotonic and v a constant, so that c is a radial geodesic joining two concentric spherical shells around q.*

Proof. Let $\alpha(s,t) = \exp_q(sv(t))$. Then $c(t) = \alpha(u(t),t)$. Now $\frac{dc}{dt} = \frac{\partial \alpha}{\partial u} u' + \frac{\partial \alpha}{\partial t}$.
Since $\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\rangle = 0$ (Gauss lemma) and $\|\frac{\partial \alpha}{\partial s}\| = 1$ (this is true at $s = 0$ and hence true for all s because geodesics preserve length), we see that $\|c'\|^2 \geq |u'|^2$ with equality if and only if $\frac{\partial \alpha}{\partial t} = 0$ and hence $v'(t) = 0$. This is easily seen to complete the proof. $\square$