NOTES FOR 25 JAN (THURSDAY)

1. Recap

(1) Proved that there are neighbourhoods (W_p, ϵ_p) such that every around every point $q \in W_p$, the exponential map diffeomorphically maps the epsilon ball to an open set. Moreover, any two points can be connected by a geodesic of length $< \epsilon_p$.
(2) Defined the geodesic normal coordinates and proved that they are normal coordinates (the metric is standard upto the first order). Discussed that they are usually not standard upto higher orders because of curvature.
(3) Proved the Gauss lemma (radial geodesics are perpendicular to geodesic spheres) and used it to prove that among some curves, geodesics have the shortest length.

2. Riemannian manifolds and metrics on vector bundles

Now we have a local length minimising property.

Corollary 2.1. *Let W_p, ϵ_p be as in one of the earlier theorems. Let $\gamma : [0,1] \to M$ be the geodesic of length $< \epsilon$ joining $q, q^{'} \in W$ and let $c : [0,1] \to M$ be any piecewise smooth path from q to $q^{'}$. Then $L(\gamma) \leq L(c)$ with equality holding if and only if c is a reparametrisation of γ.*

Proof. We can assume that $q^{'} = \exp_q(rv)$ (otherwise break c up into smaller pieces). For $\delta > 0$, the path c must contain a segment which joins the spherical shell of radius δ to the spherical shell of radius r and lies between them. So by the previous corollary, the length of this segment is at least $r - \delta$. So the length of $c \geq r$. Hence c must be a reparametrisation of γ for equality to hold. □

A piece of terminology - A normal neighbourhood is the image of an open set V under the exponential map on which it is a diffeomorphism. A totally normal neighbourhood is a normal neighbourhood such that every two points can be connected by a unique length-minimising geodesic. A geodescally convex neighbourhood is a totally normal neighbourhood where the length-minimising geodesic stays completely within the neighbourhood.

Theorem 2.2. *If $p \in M$, there exists a geodesic ball $B_{\epsilon_p}(p)$ such that every two points in the ball can be connected by a unique length minimising geodesic lying in the ball and such that the exponential map is a diffeomorphism restricted to the ball. Such a ball is called a geodesically convex ball.*

Proof. Of course there is a ball B (by the above) such that any two points can be connected by length-minimising geodesic of length $< \tilde{\epsilon}_p$. The question is whether the geodesic lies completely within the ball or not. Now we need the following lemma.

Lemma 2.3. *For any $p \in M$, there is a number $c_p > 0$ such that any geodesic in M tangent at $q \in M$ to the geodesic sphere $S_r(p)$ of radius $r < c$, stays out of the geodesic ball $B_r(p)$ for some neighbourhood of q.*

Proof. Basically we take the distance F from p to a point on the geodesic and show that F achieves a strict local minimum at q by calculating the second-derivative. This will do the job.

Let W be a totally normal neighbourhood of p. By rescaling we may assume that all the geodesics have speed 1. Let $\gamma(t,q,v) : I = (-\epsilon,\epsilon)\times TW \to M$ be the geodesic that starts at q and points initially along v. Define $u(t,q,v) = \exp_p^{-1}(\gamma(t,q,v))$. Define $F : I \times TW \to \mathbb{R}$ as $F(t,q,v) = |u(t,q,v)|^2$. Note that $\frac{\partial F}{\partial t} = 2\langle \frac{\partial u}{\partial t}, u\rangle$ and $\frac{\partial^2 F}{\partial t^2} = 2\langle \frac{\partial^2 u}{\partial t^2}, u\rangle + 2|\frac{\partial u}{\partial t}|^2$.

Suppose r is small enough so that $\exp_p(B_r(0)) \subset W$. If γ is tangent to the geodesic sphere at $q = \gamma(0,q,v)$, then by the Gauss lemma, $\frac{\partial F}{\partial t}(0,q,v) = 0$. Now note that for $q = p$, $u = tv$ and hence $\frac{\partial^2 F}{\partial t^2}(0,p,v) = 2$. Thus there is a neighbourhood V of p where this second derivative is positive. Let c be such that $\exp_p(B_c(0)) \subset V$. This will do the job. $\square$

Let c_p be the number as in the lemma. Choose (W,δ) to be a totally normal neighbourhood of p (i.e. W is totally normal and the δ ball around every point in W is totally normal) and such that $\delta < \frac{c_p}{2}$. Take $\beta < \delta$ such that $B_\beta(p) \subset W$. We shall prove that $B_\beta(p)$ is convex. If $q_1, q_2 \in \bar{B_\beta}(p)$ be connected by the unique length-minimising geodesic of length $< 2\delta < c$. This curve $\gamma \in B_c(p)$. Indeed $d(p,\gamma(t)) \leq d(p,q_1) + d(q_1,\gamma(t)) < \delta + \delta < c$ (assuming q_1 is closer to $\gamma(t)$ than q_2). If the interior of γ is not in $B_\beta(p)$, then there is a point m in the interior of γ which is at the maximum distance r from p. The points of γ in a neighbourhood of m remain in the closure of $B_r(p)$. This contradicts the above lemma. $\square$

In fact, using this (geodesic triangles) one can prove that any surface can be triangulated, i.e., piecewise linearly diffeomorphic to a simplicial complex.

Using the above corollary, we can determine the geodesics of some manifolds. Before that, we introduce a definition : If (M, g_M), (N, g_N) are two Riemannian manifolds, then a smooth diffeomorphism $f : M \to N$ is called a Riemannian isometry if $\langle v, w\rangle_{g_M(p)} = \langle f_{*p}(v), f_{*p}(w)\rangle_{g_N(f(p))}$ for all $p \in M$, i.e., $f^* g_N = g_M$. There is a theorem called the Myers-Steenrod theorem saying that a metric space isometry between two connected Riemannian manifolds is actually a Riemannian isometry. (The converse is easy by observing that if $c : [0,1] \to M$ is a smooth curve, then $l(c) = l(f \circ c)$ where f is an isometry and that if c is a geodesic then so is $f \circ c$.)

For example, reflection through a plane passing through the origin $E_2 \subset \mathbb{R}^{n+1}$ is an isometry $I : S^n \to S^n$.

1. S^2 : The fixed point set of I is a great circle $C = S^2 \cap E^2$. Take two points $p, q \in C$. If there is a unique geodesic C' connecting p, q then $I(C')$ is the unique geodesic connecting $I(p) = p$ and $I(q) = q$. Thus $C' = I(C')$ and hence $C' \subset C$. Therefore, C is a geodesic. Since there is a great circle through every point and every direction, all the geodesics of S^n are great circles. Note that a portion larger than a semi-circle is not of minimal length even among nearby paths. So geodesics stop minimising after some time.
2. $Z = S^1 \times \mathbb{R}$ (the right circular infinite cylinder) : The geodesics are $\{p\} \times L$, $S^1 \times \{q\}$, and helices. Indeed, if L is a generating line, then $I : Z - L \to \mathbb{R}^2$ given by rolling Z onto $\mathbb{R}^2$ is an isometry. (This proves that these are the only geodesics.)

Now we wind up these things with the discussion of a very important concept : A Riemannian manifold (M,g) is called geodesically complete if every geodesic $\gamma : [a,b] \to M$ can be extended to a geodesic from $\mathbb{R}$ to M. We have the very important Hopf-Rinow theorem.

Theorem 2.4. *Suppose (M,g) is a connected Riemannian manifold. Then the following are equivalent.*

1. *M is geodesically complete.*
2. *(M,d) is a complete metric space.*
3. *A set $K \subset M$ is compact if and only if it is closed and bounded.*

(4) *There is a smooth exhaustion function* $\psi : M \to \mathbb{R}$ *(i.e.* $\psi^{-1}(-\infty, c)$ *is relatively compact in* M*) such that* $|d\psi|_g \leq 1$.

(5) *There exists an exhaustive sequence* K_ν *of compact subsets of* M *(i.e.* $K_\nu \subset Int(K_{\nu+1})$, $\cup_\nu K_\nu = M$*) and functions* $\psi_\nu : M \to \mathbb{R}$ *such that* $\psi_\nu = 1$ *in a neighbourhood of* K_ν, $Supp(\psi_\nu) \subset Int(K_{\nu+1})$, $0 \leq \psi_\nu \leq 1$ *and* $|d\psi_\nu|_g \leq 2^{-\nu}$.

Moreover, any two points in a geodesically complete manifold can be joined by a minimal length geodesic.