NOTES FOR 2 JAN (TUESDAY)

1. Logistics

(1) Webpage : http://math.iisc.ac.in/~vamsipingali/339_2018.html
(2) HW - 25%, Midterm - 25%, Final - 50 % (The HW will be put up on the webpage)
(3) Office - N -23. Office hours - Wed - 4-5.
(4) Prereqs - A first course on manifolds, some analysis (Fourier analysis and a little bit of function spaces and measure theory), multivariable calculus, and functional analysis (up to and including the spectral theorem for compact self-adjoint operators). The functional analysis part can be read from the appendix in Evans' book.

2. Goals of the course (famous last words)

(1) How to write PDE on manifolds. This will include a crash course on basic Riemannian geometry. (What is a PDE on $\mathbb{R}^n$? It is an equation of the form $F(u, \nabla u, D^2 u, \ldots) = 0$. If you want to write it on a manifold, what is $D^2 u$? If you use coordinates to define it, then changing coordinates gives a different PDE. So we need some additional structure to even write PDE on manifolds. The structure we will use is a Riemannian metric.)
(2) How to prove existence and uniqueness of solutions for linear elliptic (and hopefully some nonlinear) PDE.
(3) Why you should care about studying PDE on manifolds. (Hopefully, some Hodge theory and the uniformisation theorem.)

3. The Poisson ODE and Fourier analysis

Suppose we want to solve the ODE $u^{''} = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function, then

$$u^{'}(x) = u^{'}(0) + \int_0^x f(t)dt \tag{3.1}$$

$u(x) = u(x + 2\pi)$ implies that $u^{'}(x) = u^{'}(x + 2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$). Thus $\int_0^{2\pi} f(t)dt = 0$. This is a necessary and sufficient (by the periodicity of f, $\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt$) condition. (Smoothness is guaranteed by the fundamental theorem of calculus.)

In other words, there is a unique-upto-a-constant smooth periodic solution of the ODE if and only if f is smooth, periodic, and satisfies $\int_0^{2\pi} f(t)dt = 0$. Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^∞, the map $T : C^\infty \to C^\infty$ given by $T(u) = u^{''}$ has kernel precisely the constants. Moreover, equipping this vector space with the inner product $\langle u, v\rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$. This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.

The above mentioned observations are not coincidences. Later on, we will see that many PDE (the so-called elliptic PDE) satisfy similar properties. However, to prove such things, we cannot rely on a direct formula for the solution unlike the case of ODE. So we need a more abstract, theoretical method.

Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f. Then we see that

$$\hat{u}(k)k^2 = -\hat{f}(k) \tag{3.2}$$

In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$. In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant. Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds, if the high-frequency Fourier components are "small", then the function is very "smooth" (melodious notes are not too shrill). Since $\hat{u}(k) = \frac{\hat{f}(k)}{k^2}$, u behaves more smoothly than f does. So if f is a smooth function, we expect u to be so as well.

To make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though. For example, there exist continuous functions whose Fourier series do not convergence pointwise at some points. [1] Nonetheless, we have the following useful results.

(1) Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.
(2) Parseval-Plancherel : The Fourier series transform is an isometric isomorphism between $L^2([0, 2\pi])$ and l^2.
(3) Let $C^{0,\alpha}$ $(0 < \alpha < 1)$ consist of all Hölder continuous 2π-periodic functions g, i.e., periodic functions g such that $|g(x) - g(y)| \leq C|x - y|^\alpha$ for all x, y. Note that if f is in C^1, then f is Hölder continuous.
Theorem : If $f \in C^{0,\alpha}$ then $|\hat{f}(k)| \leq \frac{K}{|k|^\alpha}$ $\forall$ $|k| \geq 1$.

Proof.

$$2\pi\frac{\widehat{f(x+h)}(k) - \hat{f}(k)}{h^\alpha} = \int_0^{2\pi} \frac{f(x+h) - f(x)}{h^\alpha} e^{-ikx}dx$$

$$\Rightarrow \left|\frac{\widehat{f(x+h)}(k) - \hat{f}(k)}{h^\alpha}\right| \leq C \tag{3.3}$$

Now

$$\left|\int_0^{2\pi} \frac{f(x+h) - f(x)}{h^\alpha} e^{-ikx}dx\right| = \left|\int_h^{2\pi+h} \frac{f(y)}{h^\alpha} e^{-ik(y-h)} - \int_0^{2\pi} \frac{f(x)}{h^\alpha} e^{-ikx}dx\right|$$

$$= \left| - \int_0^h \frac{f(y)}{h^\alpha} e^{-ik(y-h)}dy + \int_{2\pi}^{2\pi+h} \frac{f(y)}{h^\alpha} e^{-ik(y-h)}dy + \frac{1}{h^\alpha}\int_0^{2\pi} e^{-ikx} f(x)(e^{ikh} - 1)dx\right|$$

$$= \left|\frac{1}{h^\alpha}\int_0^{2\pi} e^{-ikx} f(x)(e^{ikh} - 1)dx\right| = |\hat{f}(k)|\frac{|e^{ikh} - 1|}{h^\alpha} \tag{3.4}$$

Take $h = \frac{1}{k}$. Using 3.3 and 3.4 we see that $|\hat{f}(k)| \leq \frac{K}{k^\alpha}$.

As for uniform convergence, □

[1] Already this is beginning to hint that expecting results like "If f is C^k, then u is C^{k+2}" is a bad idea from the Fourier-analytic point of view. In fact for PDE, this expectation is false.

(4) Theorem : If $f \in C^{0,\alpha}$, the Fourier series converges uniformly to f.

(5) Theorem : If $f \in C^1$ then $\hat{f'} = ik\hat{f}(k)$. This holds for higher derivatives too.

Proof.

$$\hat{f'} = \frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-ikx}dx = -\frac{1}{2\pi}\int_0^{2\pi} f(x)(e^{-ikx})'dx = \frac{1}{2\pi}\int_0^{2\pi} ikf(x)e^{-ikx}dx = ik\hat{f}(k) \tag{3.5}$$

□

(6) Theorem : If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof. If f is smooth, then $\hat{f^{(l)}}(k) = (ik)^l\hat{f}(k)$. Since $\hat{f^{(l)}}(k)$ is bounded, $\hat{f}(k)$ is rapidly decaying. By one of the earlier theorems, the convergence is uniform.

If $|a_k| \leq C_l|k|^{-l}$, then by the Weierstrass M-test (choosing $l > 1$), we see that $\sum a_k e^{ikx}$ converges uniformly to a continuous function u. The same argument also shows that $\sum (ik)^l a_k e^{ikx}$ converges uniformly to u_l. It is easy to see (fundamental theorem of calculus and interchange of summation and integration) that $u_l = u^{(l)}$. □