NOTES FOR 30 JAN (TUESDAY)

1. Recap

(1) Proved the existence of geodesically convex balls and proved that geodesics are locally length minimising.
(2) Defined isometries and stated the Myers-Steenrod theorem. Calculated geodesics for some examples.
(3) Defined geodesic completeness.

2. Riemannian manifolds and metrics on vector bundles

Theorem 2.1. *Suppose (M, g) is a connected Riemannian manifold. Then the following are equivalent.*

(1) *M is geodesically complete.*
(2) *(M, d) is a complete metric space.*
(3) *A set $K \subset M$ is compact if and only if it is closed and bounded.*
(4) *There is a smooth exhaustion function $\psi : M \to \mathbb{R}$ (i.e. $\psi^{-1}(-\infty, c)$ is relatively compact in M) such that $|d\psi|_g \leq 1$.*
(5) *There exists an exhaustive sequence K_ν of compact subsets of M (i.e. $K_\nu \subset Int(K_{\nu+1})$, $\cup_\nu K_\nu = M$) and functions $\psi_\nu : M \to \mathbb{R}$ such that $\psi_\nu = 1$ in a neighbourhood of K_ν, $Supp(\psi_\nu) \subset Int(K_{\nu+1})$, $0 \leq \psi_\nu \leq 1$ and $|d\psi_\nu|_g \leq 2^{-\nu}$.*

Moreover, any two points in a geodesically complete manifold can be joined by a minimal length geodesic.

Proof. Zeroethly, we recall that the metric d induces the same topology as the original one on the manifold. In particular, d is continuous.

First we prove that 1 implies that any two points can be joined by a minimal geodesic. Suppose $p, q \in M$ with $d(p, q) = r > 0$. Choose U_p as in an earlier theorem. Let $S \subset U_p$ be the spherical shell of radius $\delta < \epsilon$. There is a point $p_0 = \exp_p(\delta v)$, $\|v\| = 1$ on S such that $d(p_0, q) \leq d(s, q)$ for all $s \in S$ (by compactness of S). We claim that $\exp_p(rv) = q$ (thus proving what we want). To prove this we prove that $d(\gamma(t), q) = r - t$ for $t \in [\delta, r]$.

Indeed, firstly, since every curve from p to q must intersect S (Intermediate value theorem : $d(p, p) = 0, d(p, q) > \delta$) we see that $d(p, q) = \min_{s \in S}(d(p, s) + d(s, q)) = \delta + d(p_0, q)$. Thus $d(p_0, q) = r - \delta$.

The set A of $t \in [\delta, r]$ satisfying the property that $d(\gamma(t), q) = r - t$ is non-empty. It is closed by continuity. If we prove that it is open, then by connectedness we will be done. If $s_0 < r \in A$. Let $\delta^{'}$ be small. We want to show that $\delta^{'} + s_0 \in A$. Let $B_{\delta^{'}}(\gamma(s_0))$ be the geodesically convex ball with boundary $S^{'}$. Let $x_0^{'}$ be the point of minimum of $d(x, q)$ on $S^{'}$.

Note that $r - s_0 = d(\gamma(s_0), q) \leq \delta^{'} + d(x_0^{'}, q)$. Actually, since there is a curve γ_ϵ joining $x_0^{'}, q$ such that $d(x_0^{'}, q) \leq l(\gamma_\epsilon)_{x_0^{'}}^{q} \leq d(x_0^{'}, q) + \epsilon$, we see that the curve consisting of the geodesic between $s_0, x_0^{'}$ of length $\delta^{'}$ and then γ_ϵ joining $x_0^{'}, q$ is of length at most $\delta^{'} + d(x_0^{'}, q) + \epsilon$. Thus $d(\gamma(s_0), q) = \delta^{'} + d(x_0^{'}, q)$ (it is less than this quantity and there are curves whose length is arbitrarily close to this quantity).

Now $d(p, x_0^{'}) \geq d(p, q) - d(q, x_0^{'}) = s_0 + \delta^{'}$. By the local minimising properties of geodesics, it is clear that $\gamma(s_0 + \delta^{'}) = x_0^{'}$. Thus $d(\gamma(s_0 + \delta^{'}), q) = r - (s_0 + \delta^{'})$.

(1) $1 \Rightarrow 3$: Of course compact sets are closed and bounded. Suppose K is closed and bounded. Then there is a metric ball $K \subset B_R(p)$ where $p \in K$. By existence of length minimising geodesics, there exists a ball $B_r(0) \in T_pM$ such that $B_R(p) \subset \exp(\bar{B}_r(0))$. Since the right hand side is compact, so is K.

(2) $3 \Rightarrow 2$: If x_n is a Cauchy sequence, then the set K = closure of $\{x_1, x_2, \ldots\}$ is certainly closed and bounded. Thus it is compact. Hence x_n has a convergent subsequence. Therefore x_n converges.

(3) $2 \Rightarrow 1$: Suppose (M, g) is not geodesically complete. Then there is an arc-length parametrised geodesic γ defined on $s < s_0$ but not at s_0. Let $s_n < s_0 \to s_0$. For large n, m, $d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m| < \epsilon$. Hence $\gamma(s_n)$ is Cauchy and converges to p_0. Let W_{p_0}, δ be a geodesically convex normal neighbourhood around p_0. Assume that n, m are large so that $|s_m - s_n| < \delta$. There is a geodesic α connecting $\gamma(s_n), \gamma(s_m)$ with length less than δ. Wherever γ is defined, $\alpha = \gamma$ (by uniqueness). Suppose β is the geodesic connecting $\gamma(s_N)$ and p_0. Let M be so large that $d(\gamma(s_m), p_0) \leq \epsilon$ for all $m \geq M$. Now $L(\gamma)_{s_N}^{s_0} = |s_N - s_0| = \lim_{m\to\infty} |s_N - s_m| = \lim_{m\to\infty} d(\gamma(s_N), \gamma(s_m)) \leq \lim_{m\to\infty}(d(\gamma(s_N), p_0)) + \epsilon$. Hence $L(\gamma)_{s_N}^{s_0} \leq d(\gamma(s_N), p_0) + \epsilon \ \forall \epsilon$. Thus means that $\gamma = \beta$. Thus β extends γ past s_0 to $s_0 + \delta$.

(4) $2 \Rightarrow 4$: Fix $x_0 \in M$. Set $\psi_0(x) = \frac{1}{4} d(x_0, x)$. Of course $\psi_0(x)$ is smooth when x is close to x_0. In fact, it is Lipshitz with constant $\frac{1}{4}$. Since Lipshitz functions are differentiable almost everywhere (Rademacher's theorem), ψ_0 is so and $|d\psi_0| \leq \frac{1}{4}$ almost everywhere. ψ_0 is of course an exhaustion except that it is not smooth. If we manage to smooth it out to ψ satisfying $|\psi - \psi_0| \leq 1$ and $|d\psi| \leq 1$, we will be done. This smoothing will be given as a HW exercise. (Essentially, you take a locally finite cover by normal balls of a small enough size so that the metric is close to being Euclidean. Let ρ_α be a partition-of-unity. Now using mollification, you can approximate $\rho_\alpha \psi_0$ by a smooth function ψ_α uniformly and let $\psi = \sum_\alpha \psi_\alpha$. You need to be clever in approximating so that the derivative of ψ is not too big.)

(5) $4 \Rightarrow 5$: Choose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho = 1$ on $(-\infty, 1.1)$ and $\rho = 0$ on $[1.9, \infty)$, $0 \leq \rho^{'} \leq 2$ on $[1, 2]$. Then $K_\nu = \{x : \psi(x) \leq 2^{\nu+1}\}$ and $\psi_\nu(x) = \rho(2^{-\nu-1}\psi(x))$.

(6) $5 \Rightarrow 4$: Set $\psi = \sum 2^\nu (1 - \psi_\nu)$.

(7) $4 \Rightarrow 3$: The inequality $|d\psi|_g \leq 1$ implies that $|\psi(x) - \psi(y)| \leq d(x, y)$ and hence all finite closed balls are compact. (Indeed, if x_k in a closed ball B_R is sequence without a convergent subsequence, then $\psi(x_k) \to \infty$ because ψ is proper. But $d(x_k, y)$ is bounded.)

□

Here are examples of complete manifolds :

(1) Compact manifolds. In fact, as a corollary of the Hopf-Rinow theorem, we see that there is always a minimal geodesic joining any two points on a compact manifold.
(2) $\mathbb{R}^n$
(3) $\mathbb{H}^n$. Indeed, one can write down the geodesics (they are vertical lines and semicircles) explicitly to see this.

3. Connections and curvature

Here are a bunch of observations / questions :

(1) The Christoffel symbols $\Gamma^i_{jk} = F(g, \partial g)$ do *not* transform like a tensor does upon change of coordinates. (In english, this means, when I change coordinates, the new symbols are not linearly dependent on the old ones.) This can be verified easily by calculation. So what kind of objects are they ?

(2) In g.n.c, the second term in the Taylor expansion of the metric was a tensor (which we called the Riemann curvature tensor) depending on two derivatives of the metric. Since this seems to detect "curvature" in the english sense of the word (for example, it is 0 when the metric can be made Euclidean), we need to see if there is a more invariant way to define this quantity.

(3) If one wants to follow the shortest path in a city, one follows "one's nose", i.e., the tangent vector of the path is transported "parallel" to itself, i.e., there is no acceleration in any other direction. So we need to define a notion of "parallel transport" along a curve.