NOTES FOR 3 APR (TUESDAY)

1. Recap

(1) Proved that elliptic operators are Fredholm (from $H^l \to L^2$).
(2) For coercive $2o$ order operators we proved existence of weak solutions.

2. Elliptic operators - Diagonalisability

Define the operator $f \in L^2 \to u_f \in H^o \subset L^2$. This is a compact operator.

Lemma 2.1. *The operator $K(f) = u_f$ is self-adjoint.*

Proof.

$$(v, Kf) = B[Kv, Kf] = B[Kf, Kv] = (f, Kv) \tag{2.1}$$

□

Hence by the spectral theorem, its spectrum consists only of countably many eigenvalues, each eigenspace is finite dimensional, and its eigenvalues are bounded above with 0 as the only accumulation point and its eigenvectors span all of L^2. Moreover, $K - \mu I$ is an isomorphism unless μ is an eigenvalue. Also, by Fredholm theory, $(K - \mu I)u = f$ has a solution if and only if f is orthogonal to the kernel of $K - \mu I$. Now $Au - \lambda u = f$ has a solution if and only if u solves $Au = \lambda u + f$, i.e., $u = K(\lambda u + f)$ and hence if and only if f is orthogonal to the kernel of $Au - \lambda u$ which is the kernel of Lu (because of regularity). Also, the spectrum of L consists only of eigenvalues (going off to ∞) such that the eigenspaces are finite dimensional and span all of L^2.

In the case of the Hodge Laplacian, in normal coordinates one can easily see that $\Delta_d = \nabla^* \nabla + \textit{lower order terms}$. Thus, $(\Delta_d u, u) = (\nabla u, \nabla u) + (loweru, u)$. Now $|(loweru, u)| \leq C \|\nabla u\|_{L^2} \|u\|_{L^2} \leq \frac{1}{2}\|\nabla u\|^2_{L^2} + C_1 \|u\|^2_{L^2}$. Hence, $(\Delta_d u, u) + (C_1 + \frac{1}{2})(u, u) \geq \frac{1}{2}(\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2})$. This proves the Garding coercivity inequality and hence the Hodge theorem. (Actually, all we need is the diagonalisability part because the rest of Hodge follows from the Fredholmness of the elliptic operator Δ_d.)

Actually, since lower order terms do not make a difference to Fredholmness, this also proves that the above kind of operators plus lower order terms are still Fredholm.

3. Schauder and $W^{k,p}$ estimates

As in the case of H^s and $C^{k,\alpha}$ we can define the $W^{k,p}(M, E)$ spaces either globally using connections (using either weak derivatives or as the completion of smooth sections) or by partitions-of-unity and the local definition.

If L is elliptic (with smooth coefficients) and $u \in L^p$ is a distributional solution of $Lu = f$ where $f \in W^{k,p}$, then $u \in W^{k+o,p}$ with $\|u\|_{W^{k+o,p}} \leq C_{k,p}(\|f\|_{W^{k,p}} + \|u\|_{L^p})$. Likewise, if $f \in C^{k,\alpha}$ and $u \in C^{o,\alpha}$ is a solution, then $u \in C^{k+o,\alpha}$ with $\|u\|_{C^{k+o,\alpha}} \leq C_{k,\alpha}(\|f\|_{C^{k,\alpha}} + \|u\|_{C^0})$ (the Schauder estimates). (These things are in L. Nicolaescu's lectures on the geometry of manifolds.) We shall not prove these results. The Schauder estimates are not too hard to prove but the $W^{k,p}$ estimates require some heavy harmonic analysis (the Calderon-Zygmund inequality).

We prove a small special related result just to see what the proofs look like. (This is from Gilbarg-Trudinger.)

Lemma 3.1. *Let $B(r)$ be the open r-ball in $\mathbb{R}^m$ and $\bar{B}(r)$ be the closed one. Suppose $\Gamma(x) = \frac{1}{2\pi}\ln(|x|)$ if $n = 2$ and $\Gamma(x) = \frac{1}{n(2-n)\omega_n}|x|^{2-n}$ if $n > 2$ (where ω_n is the volume of the unit ball). Then the function $u(x) = \displaystyle\int_{\bar{B}(1)} \Gamma(x-y)f(y)dy$ is in $C^2(\bar{B}(r))$ ($r < 1$) if $f \in C^{0,\alpha}(\bar{B}(1))$, and $\Delta u = f$.*

Proof. See Gilbarg-Trudinger (chapter 4). □

Actually, the Schauder estimates hold under weaker assumptions : If L is a second order operator with $C^{0,\alpha}$ coefficients on Ω acting on functions (as opposed to sections), then the (interior) Schauder estimates still hold. This sort of a thing is extremely useful. For instance, if we have a $C^{2,\alpha}$ solution such that $D^2u > 0$ of $\det(D^2u) = f$ on Ω where f is smooth, then u is actually smooth. Indeed, differentiate and use the Schauder theory to bootstrap.