NOTES FOR 6 MAR (TUESDAY)

1. Recap

(1) Proved the divergence theorem
(2) Defined the Hodge star and proved its properties.

2. Divergence, Stokes' theorem, and Laplacians

Definition 2.1. Let α be a smooth k-form. Then $d^{\dagger}\alpha = (-1)^{m(k+1)+1} * d * \alpha$. Thus $d^{\dagger}\alpha$ is a smooth $k-1$-form depending on the first derivative of α (it is a first order differential operator).

The "codifferential" satisfies the following properties :

(1) $d^{\dagger} f = 0$ where f is a smooth function.
(2) $d^{\dagger} \circ d^{\dagger} = 0$.
(3) $(d\alpha, \beta) = \int_M \langle d\alpha, \beta \rangle vol_g = \int_M \langle \alpha, d^{\dagger}\beta \rangle vol_g = (\alpha, d^{\dagger}\beta)$. Thus, $d^{\dagger}$ is formally speaking, the adjoint of d.
(4) If X is a vector field and ω_X is the dual 1-form, then $d^{\dagger}\omega_X = -div(X)$. Hence, $d^{\dagger}df = -\Delta f$.

Proof. (1) Obvious because f is a 0-form.
(2) $d^{\dagger} \circ d^{\dagger} = \pm * d *^2 d* = \pm * d \circ d* = 0$
(3) Suppose β is a k-form and α a $k-1$ form.

$$(\alpha, d^{\dagger}\beta) = \int_M \alpha \wedge (-1)^{m(k+1)+1} * *d * \beta = \int_M \alpha \wedge (-1)^{m(k+1)+1+(m-k+1)(m-(m-k+1))} d * \beta$$
$$= \int_M (-1)^k (-1)^k (d(\alpha \wedge *\beta) - d\alpha \wedge *\beta) = \int_M d\alpha \wedge *\beta = (d\alpha, \beta) \tag{2.1}$$

(4) Note that

$$(d^{\dagger}\omega_X, f) = (\omega_X, df) = \int_M g^{ij} (\omega_X)_i \frac{\partial f}{\partial x^j} vol$$
$$= \int_M g^{ij} g_{ik} X^k \frac{\partial f}{\partial x^j} vol = (X, \nabla f) = -(div(X), f)$$
$$\Rightarrow (d^{\dagger}\omega_X + div(X), f) = 0 \ \forall \ f \in C^{\infty}(M) \tag{2.2}$$

The last equality implies the result because we can choose f to be a mollifier supported inside a coordinate chart and take limits. □

The last equality motivates us to make the following definition :

Definition 2.2. Suppose α is a smooth k-form on a compact oriented Riemannian manifold (M, g). Define the second order linear partial differential operator (the Hodge Laplacian or the Laplace-Beltrami operator) as the k-form $\Delta_d \omega = (dd^{\dagger} + d^{\dagger}d)\omega$.

Let us calculate this on $\mathbb{R}^m$ with the Euclidean metric and the usual orientation. (Remember that this Laplacian depends on the choice of a metric and an orientation.) Let $\eta = \eta_I dx^I$ be a k-form (where the sum is over all indices, whether increasing or not).

$$d\eta = \frac{\partial \eta_I}{\partial x^j} dx^j \wedge dx^I$$
$$d^\dagger \eta = (-1)^{m(k+1)+1} * d * \eta = (-1)^{m(k+1)+1} * d(\eta_I (-1)^{sgn(I)} dx^{I^c})$$
$$= (-1)^{m(k+1)+1+sgn(I)} * \frac{\partial \eta_I}{\partial x^j} dx^j \wedge dx^{I^c} = (-1)^{m(k+1)+1+sgn(I)} \frac{\partial \eta_I}{\partial x^j} (-1)^{sgn(j,I^c)} dx^{i_1} \ldots dx^{i_{a_j(I)-1}} \wedge \hat{dx^j} \ldots$$
$$= (-1)^{m(k+1)+1+sgn(I)+m-k+a_j(I)-1+sgn(I^c)} \frac{\partial \eta_I}{\partial x^j} dx^{i_1} \ldots \wedge \hat{dx^j} \ldots$$
$$= (-1)^{m(k+1)+m-k+a_j(I)+k(m-k)} \frac{\partial \eta_I}{\partial x^j} dx^{i_1} \ldots \wedge \hat{dx^j} \ldots = (-1)^{a_j(I)} \frac{\partial \eta_I}{\partial x^j} dx^{i_1} \ldots \wedge \hat{dx^j} \ldots$$
$$\Delta_d \eta = (dd^\dagger + d^\dagger d)\eta = d((-1)^{a_j(I)} \frac{\partial \eta_I}{\partial x^j} dx^{i_1} \ldots \wedge \hat{dx^j}) + d^\dagger \frac{\partial \eta_I}{\partial x^k} dx^k \wedge dx^I$$
$$= \sum_{I,k,j \in (i_1,\ldots,i_k)} (-1)^{a_j(I)} \frac{\partial^2 \eta_I}{\partial x^k \partial x^j} dx^k \wedge dx^{i_1} \ldots \wedge \hat{dx^j} + \sum_{I,k,j \in (k,i_1,\ldots,i_k)} (-1)^{a_j(k,I)} \frac{\partial^2 \eta_I}{\partial x^k \partial x^j} dx^k \wedge dx^{i_1} \ldots \hat{dx^j} \ldots$$
$$= -\sum_{I,k} \frac{\partial^2 \eta_I}{\partial (x^k)^2} dx^I = -(\Delta \eta_I) dx^I$$

So, in particular, in Euclidean space, if we compute the principal symbol of the Hodge Laplacian, i.e., we replace the highest order derivatives by a vector $\vec{\zeta}$, we get $\sigma_{\Delta_d}(\vec{\zeta}) = -\begin{bmatrix} |\zeta|^2 & 0 & \ldots \\ 0 & |\zeta|^2 & \ldots \\ \vdots & \ddots & \ldots \end{bmatrix}$. Hence this operator is elliptic with constant coefficients. This holds true even for the flat torus.

Before we proceed further with the analysis of the PDE $\Delta_d \eta = \alpha$, we define a general notion of a Laplacian (the so called Bochner Laplacian or the Rough Laplacian). To do so, suppose (E, ∇, h) is a vector bundle on a compact oriented Riemannian manifold (M, g) with a metric (h) compatible connection ∇. Then we identify the formal adjoint $\nabla^\dagger : \Gamma(T^*M \otimes E) \to \Gamma(E)$ of the connection $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ defined by the property

$$(\nabla^\dagger \alpha, \beta) = \int_M \langle \nabla^\dagger \alpha, \beta \rangle_h vol_g = \int_M \langle \alpha, \nabla \beta \rangle_{g^* \otimes h} vol_g = (\alpha, \nabla \beta) \tag{2.3}$$

We need to prove that such an operator is actually a differential operator by finding a formula for it. (Such an operator is unique - Why ?) Suppose we choose an orthonormal normal trivialisation e_i for (E, ∇, h) and normal coordinates x^μ for g at p, then $A(p) = 0, h(p) = Id, g = Id + O(x^2) = g^*$. Let $\alpha = \alpha^i_\mu dx^\mu \otimes e_i$, $\beta = \beta^j e_j$. Then

$$\langle \alpha, \nabla \beta \rangle_{g^* \otimes h}(p) = \sum_{\mu,i} \alpha^i_\mu(p) \frac{\partial \beta^i}{\partial x^\mu}(p) = \sum_{\mu,i} \frac{\partial \alpha^i_\mu \beta^i}{\partial x^\mu}(p) - \frac{\partial \alpha^i_\mu}{\partial x^\mu}(p)\beta^i(p)$$
$$= div(\langle \alpha, \beta \rangle^\sharp)(p) - \frac{\partial \alpha^i_\mu}{\partial x^\mu}(p)\beta^i(p)$$

Now the expression $-\frac{\partial \alpha^i_\mu}{\partial x^\mu}(p)\beta^i(p)$ can be written as $-\langle tr(\nabla \alpha), \beta \rangle_h(p)$ which is a globally defined quantity. By the divergence theorem, $\nabla^\dagger \alpha = -tr(\nabla \alpha)$. So finally,

Definition 2.3. Suppose (M, g) is a compact oriented Riemannian manifold (without boundary as usual) and (E, ∇, h) is a vector bundle with a metric h and a metric-compatible connection ∇. The Bochner Laplacian (sometimes called the Rough Laplacian) is defined as $\nabla^{\dagger}\nabla : \Gamma(E) \to \Gamma(E)$ where $\nabla^{\dagger}\alpha = -tr(\nabla\alpha)$.

Suppose we take $E = \Omega^k(M)$, then potentially, we have two Laplacians, Δ_d and $\nabla^*\nabla$. It turns out that

$$\Delta_d \eta = \nabla^* \nabla \eta + Curvature(\eta) \tag{2.4}$$

where the last term is something that depends linearly on η with coefficients depending on the Riemann tensor. This sort of an identity relating two different Laplacians is called a Bochner-Weitzenböck identity. So, taking inner product with η and integrating,

$$(d\eta, d\eta) + (d^{\dagger}\eta, d^{\dagger}\eta) = (\nabla\eta, \nabla\eta) + (\eta, Curvature(\eta)) \geq (\eta, Curvature(\eta)) \tag{2.5}$$

So if $\Delta_d \eta = 0$, i.e., η is Harmonic, and the curvature term is positive, we have a contradiction unless $\eta = 0$. This sort of a conclusion turns out to be useful for topology. This method is called the Bochner technique for proving non-existence of non-trivial Harmonic objects.