NOTES FOR 6 FEB (TUESDAY)

1. Recap

(1) Made more observations and concluded that we need a notion of directional derivative of a section of a vector bundle.
(2) Defined that notion (of a connection) $\nabla_X s$. It obeyed three properties (Tensoriality in X, linearity in s, Leibniz rule). ∇s can also be thought of a vector-valued 1-form.
(3) Looked at $\nabla s = (d+A)\vec{s}$ locally and saw how it transformed under change of a trivialisation. (Not like a tensor.) However, every connectio n is $\nabla_0 +$ a section of $End(V) \otimes T^*M$.

2. Connections and curvature

We prove a useful little lemma here

Lemma 2.1. *If two smooth sections $s_1, s_2 : M \to V$ satisfy $s_1 = s_2$ on a neighbourhood U of p, then $\nabla s_1(p) = \nabla s_2(p)$. T(That is, the directional derivative at p depends only on local information about s near p.)*

Proof. Taking $s = s_1 - s_2$, we just have to show that $\nabla s(p) = 0$ if $s = 0$ on U. Indeed, taking a local trivialisation, $s = s^i e_i$ and hence $\nabla s(p) = (ds^i(p) + ([A](p)\vec{s}(p))^i)e_i = 0$. □

In fact, the above observation shows that in order to know ∇s at p, it is enough to take any smooth section $\tilde{s}$ on U, define $s = \rho\tilde{s}$ where ρ is a bump function vanishing outside U (and equal to 1 in a smaller neighbourhood of p), and find out $\nabla s(p)$ for all such sections.
Before going further, we prove a very very useful lemma. (This is like the existence of normal coordinates.)

Lemma 2.2. *Suppose ∇ is a connection on V. Suppose $p \in M$. There exists a trivialisation such that $A(p) = 0$ in this trivialisation.*

Proof. Choose any trivialisation in a neighbourhood U of p. Assume that (x, U) is also a coordinate chart for M such that p corresponds to $x = 0$. Let $\nabla = d + \tilde{A}$ on U. If we change the trivialisation using a transition function g, then $A = g\tilde{A}g^{-1} - dgg^{-1}$. Suppose $\tilde{A}(p) = B_i dx^i$ where B_i are real (or complex) $r \times r$ matrices. Define $g = I + x^i b_i$. For sufficiently small x, g is invertible. Now $g(p) = g^{-1}(p) = I$ and $dg = B_i dx^i = \tilde{A}(p)$. Thus $A(p) = \tilde{A}(p) - \tilde{A}(p) = 0$. □

Note that the trivial bundle $M \times \mathbb{R}^r$ has an obvious connection - the usual directional derivative. Indeed, there is a global trivialisation. Set $A = 0$ and define $\nabla s = d\vec{s}$.

Another point : If $T : V_1 \to V_2$ is a bundle isomorphism, and V_2 has a connection ∇, we can define a connection on V_1 : $(T^*\nabla)s = T^{-1}(\nabla T(s))$. This is called the pullback connection. Locally, the connection matrix of one-forms is $T^{-1}AT + T^{-1}dT$.

Theorem 2.3. *Every vector bundle has a connection.*

Proof. Suppose M is covered by a locally finite cover U_α of trivialisation open sets for V. Suppose $T_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^r$ is the trivialising isomorphism of bundles. As discussed, there is an obvious connection ∇_α on the trivial bundle $U_\alpha \times \mathbb{R}^r$. Define $\tilde{\nabla}_\alpha = T_\alpha^* \nabla_\alpha$ as a connection on V on the set U_α, i.e., if s is any local section on U_α, $\tilde{\nabla}_\alpha s$ is a section of $T^*U_\alpha \times V|_{U_\alpha}$. Suppose ρ_α is a partition-of-unity subordinate to U_α.

Now, define $\nabla s = \sum_\alpha \rho_\alpha \tilde{\nabla}_\alpha s$. The meaning of this statement is "Take s, restrict it to U_α, calculate $\tilde{\nabla}_\alpha s$ as a section over U_α, multiply by ρ_α and extend it to all of M by 0 outside U_α, and sum over all α. It is a finite sum at every point because of local finiteness of the cover. Thus we have a section of $V \otimes T^*M$"

We still have to prove that ∇ is a connection. Indeed,

$$\begin{aligned}\nabla(fs) &= \sum_\alpha \rho_\alpha \tilde{\nabla}_\alpha (fs) = \sum_\alpha \rho_\alpha (df \otimes s + f \tilde{\nabla}_\alpha s) \\ &= df \otimes s \sum_\alpha \rho_\alpha + f \nabla s = df \otimes s + f \nabla s \end{aligned} \tag{2.1}$$

□

So every vector bundle can be equipped with a way to take directional derivatives. There can be more than one way (infinitely many in fact). We can now define the notion of a "constant", rather a "parallel" section.

Definition 2.4. A smooth section s is said to be parallel with respect to a connection ∇ if it satisfies $\nabla s = 0$.

We can do better. Suppose $\gamma : [0,1] \to M$ is a smooth curve. Assume that s_0 is a vector in $V_{\gamma(0)}$.

Definition 2.5. The parallel transport of s_0 is a section s on a neighbourhood of the image of γ such that $\nabla_{\gamma'(t)} s = 0$ on the image of γ (where we are assuming that $\gamma'(t)$ has been extended arbitrarily to a smooth vector field on a smaller open subset of the neighbourhood on which s is defined).

The neighbourhood does not make any difference in the above definition. The definition locally means this : If we choose a trivialisation and a coordinate chart on the manifold, we are required to solve an ODE : $\frac{d\vec{s}}{dt} + A_{\gamma(t)}(\gamma'(t))\vec{s} = 0$ with $\vec{s}(0) = \vec{s}_0$. Of course this system of ODE has a unique smooth solution for a short period of time. In fact, it can be proven to have a solution for all time.

Now we turn to another notion arising from a connection. What if we want to take the second derivative ? There is a nice way to do this using a connection, but let us return to that later. For now, let us be very naive. Note that ∇ takes sections to vector-valued 1-forms. What if we want to apply ∇ again ? Unfortunately, unless we have a way to differentiate 1-forms, there is no meaning to differentiating $\omega \otimes s$. But we actually do have a way to differentiate 1-forms using the exterior derivative d ! So, define the following map $d^\nabla : \Gamma(V \otimes T^*M) \to \Gamma(V \otimes \Omega^2(M))$ given by $d^\nabla(\omega \otimes s) = d\omega \otimes s + \omega \wedge \nabla s$ and extending it linearly. Of course, $d^\nabla(f\omega \otimes s) = df \wedge \omega \otimes s + f d^\nabla(\omega \otimes s)$. So indeed, tensoriality holds and hence the image of d^∇ is a vector-valued 2-form. Actually, let's take this opportunity to define $d^\nabla : \Gamma(V \otimes \Omega^r M) \to \Gamma(V \otimes \Omega^{r+1} M)$ as $d^\nabla(s \otimes \omega) = \nabla s \wedge \omega + s \otimes d\omega$.

It is natural to ask whether $(d^\nabla)^2 = 0$ on sections (i.e. vector-valued 0-forms). But this is not true ! Indeed, locally, $d^\nabla s = (d\vec{s} + A\vec{s})$. Thus $(d^\nabla)^2 s = d(d\vec{s} + A\vec{s}) + A \wedge (d\vec{s} + A\vec{s}) = 0 + d(A\vec{s}) + A \wedge d\vec{s} + A \wedge A\vec{s} = dA\vec{s} - A \wedge d\vec{s} + A \wedge d\vec{s} + A \wedge A\vec{s} = (dA + A \wedge A)\vec{s} = F\vec{s}$ where F is locally a matrix of 2-forms called the curvature of (V, ∇). In other words, $(d^\nabla)^2 s$ depends linearly on s and not on any derivative of it ! More curiously, if we calculate how F changes when we change the trivialisation, we see that $\tilde{F} = gFg^{-1}$. In other words, F is actually a section of $End(V) \otimes \Omega^2(M)$. (We can do this calculation more invariantly by proving tensoriality, i.e., $(d^{\nabla})^2(fs) = f(d^{\nabla})^2 s$.)

Definition 2.6. The curvature F of a connection ∇ is a section of $End(V) \otimes \Omega^2(M)$ defined as $Fs = (d^{\nabla})^2 s$. It locally has the formula, $F = dA + A \wedge A$.

If V is a line bundle, $A \wedge A = 0$ and $F = dA$ is a global closed 2-form (because $End(L)$ is a trivial bundle).

Here is an interesting observation :

Lemma 2.7. *If (L, ∇) is a (real or complex) line bundle, then its curvature F is a globally defined closed 2-form whose De Rham cohomology class is independent of the connection chosen.*

Proof. We already saw that F is a globally defined close 2-form. Suppose $\nabla_1, \nabla_2 = \nabla_1 + a$ are two connections where a is a section of $End(L) \otimes T^*M$. Noting that $End(L)$ is trivial, a is a globally defined 1-form. Now $F_2 = dA_2 = dA_1 + da = F_1 + da$. Therefore $[F_2] = [F_1]$. □