NOTES FOR 8 MAR (THURSDAY)

1. Recap

(1) Defined the codifferential $d^\dagger$, the Hodge Laplacian, and the Rough Laplacian.
(2) Proved a formula for the Hodge Laplacian on $\mathbb{R}^n$ with the usual orientation and the Euclidean metric.

2. Statement of the Hodge theorem and applications

To calculate the De Rham cohomology groups $H^k(M)$, it is useful to have "good, canonical" representatives of each cohomology class. That is, given a class $[\eta]$ consisting of forms $\alpha = \eta + d\gamma$, we want to find the "best" possible $\alpha \in [\eta]$. More precisely, we want to minimise the "energy" $E_g(\alpha) = \displaystyle\int_M |\alpha|_g^2 vol_g$. Suppose α is such a smooth minimiser. Then $\frac{dE(\alpha+td\gamma)}{dt}|_{t=0} = 0\ \forall\ \gamma$.

$$\frac{dE(\alpha+td\gamma)}{dt}|_{t=0} = \int_M 2\langle d\gamma, \alpha\rangle = 2(d\gamma,\alpha) = 0 \Leftrightarrow (\gamma, d^\dagger\alpha) = 0 \Leftrightarrow d^\dagger\alpha = 0 \tag{2.1}$$

This means that $d\alpha = 0 = d^\dagger\alpha \Rightarrow \Delta_d\alpha = 0$. Conversely, if $\Delta_d\alpha = 0$, then taking inner product with α we see that $\|d\alpha\|^2 + \|d^\dagger\alpha\|^2 = 0$. This means that $d\alpha = 0 = d^\dagger\alpha$. So, ideally, we'd like a statement to the effect of:

Theorem 2.1 (Hodge's theorem)**.** *Suppose (M,g) is compact and oriented. The space of Harmonic forms $\mathcal{H}^k$ among the space of smooth forms is finite dimensional. Therefore there is an orthogonal projection $H : Smooth\ k\ forms \to \mathcal{H}^k$ and a unique operator $G : Smooth\ k\ forms \to Smooth\ k\ forms$ such that $Gd = dG, d^\dagger G = Gd^\dagger$ and $I = H + \Delta_d G$. As a consequence, every De Rham cohomology class has a unique harmonic representative. Also, the Hodge Laplacian is diagonalisable, i.e., there is a complete orthonormal basis of eigenvectors.*

If we manage to prove this, we have some wonderful conclusions (for compact oriented manifolds):

(1) A weak form of Poincaré duality : The map $H^k(M) \times H^{m-k}(M) \to \mathbb{R}$ given by $[\omega],[\eta] \to \displaystyle\int_M [\omega\wedge\eta]$ is non-degenerate. Thus $dim(H^k(M)) = dim(H^{m-k}(M))$. Indeed, choose any metric on M and suppose $\omega \in [\omega]$ is harmonic, i.e., $d\omega = d^\dagger\omega = 0$. Then $*\omega$ is also harmonic because $d*\omega = \pm**d*\omega = 0$ and $d^\dagger*\omega = \pm*d*^2\omega = \pm*d\omega = 0$. Now $\displaystyle\int \omega\wedge*\omega = \|\omega\|^2 = 0$ if and only if $\omega = 0$, i.e., $[\omega] = [0]$. The Poincaré duality theorem implies that $\chi(M) = dim(H^0(M)) - dim(H^1(M)) + \ldots$ is zero for odd dimensional manifolds. This $\chi(M)$ turns out to be the Euler characteristic, i.e., the alternating sum of the vertices, edges, etc if you triangulate the manifold.
(2) A weak form of the Kunneth formula : $H^k(M\times N) \simeq \oplus_{l=0}^k H^l(M)\otimes H^{k-l}(N)$ with the map being $\oplus[\omega_i]\otimes[\eta_j] \to \sum[\pi_1^*\omega_i \wedge \pi_2^*\eta_j]$. Choose metrics $g_M, g_N, g_M\times g_N$ on $M, N, M\times N$ respectively, and suppose we represent all classes with their harmonic representatives. Then

$\pi_1^*\omega_i \wedge \pi_2^*\eta_j$ are harmonic with respect to the product metric. Indeed, obviously they are closed. Now

$$\begin{aligned}(d^{\dagger_{M\times N}}\pi_1^*\omega_i \wedge \pi_2^*\eta_j, \pi_1^*\alpha \wedge \pi_2^*\beta) &= (\pi_1^*\omega_i \wedge \pi_2^*\eta_j, \pi_1^*d\alpha \wedge \pi_2^*\beta \pm \pi_1^*\alpha \wedge \pi_2^*d\beta) \\ &= (\pi_1^*\omega_i, \pi_1^*d\alpha)(\pi_2^*\eta_i, \pi_2^*\beta) + (\pi_1^*\omega_i, \pi_1^*\alpha)(\pi_2^*\eta_i, \pi_2^*d\beta) = 0\end{aligned} \tag{2.2}$$

In fact, one can prove that $\Delta_{M\times N} = \Delta_M + \Delta_N$. Thus, the map at the level of harmonic forms is well-defined. It is clear that it is injective. To prove it is surjective requires some more effort. One has to identify the eigenvectors of the Laplacian and prove it consists of decomposable forms.

Indeed, first one proves that if ω_i, η_j are the orthonormal bases of eigenvectors of Δ_M, Δ_N respectively, then $\pi_1^*\omega_i \wedge \pi_2^*\eta_j$ form a complete orthonormal basis for $L^2(M \times N, g_M \times g_N)$. This can be accomplished by proving that if α is any L^2 form, then $(\alpha, \pi_1^*\omega_i \wedge \pi_2^*\eta_j) = 0$ for all i, j, then α ought to be 0. Indeed, this implies that $(\alpha, \pi_1^*\omega \wedge \pi_2^*\eta) = 0$ for all ω, η because ω_i, η_j form bases for $L^2(M, g_M), L^2(N, g_N)$. Now near a point p, α can be written as a finite sum of such decomposable forms. Using a cut-off function, one can see that $\alpha(p) = 0\ \forall\ p$.

Second, note that $\Delta_{M\times N}(\pi_1^*\omega_i \wedge \pi_2^*\eta_j) = (\lambda_i + \mu_j)\pi_1^*\omega_i \wedge \pi_2^*\eta_j$. So $\pi_1^*\omega_i \wedge \pi_2^*\eta_j$ form a complete basis of eigenvectors for $\Delta_{M\times N}$. (A small argument shows that if you have a complete basis of eigenvectors, then every eigenvector better be one of these (or in the case of repeated eigenvalues, a linear combination of these).) Since $\lambda_i + \mu_j \geq 0$, equality holds if and only if $\lambda_i = \eta_j = 0$. Therefore, the harmonic forms of $\Delta_{M\times N}$ are obtained only this way.