NOTES FOR 9 JAN (TUESDAY)

1. Recap

(1) Solved the Poisson equation on the torus rigorously using multidimensional Fourier series. The same linear algebra intuition holds.
(2) Saw that even if f is in L^2, we can find a C^1 function u satisfying $u'' = f$ in some sense (in the sense of Fourier series). We also defined a distributional solution and saw that u satisfied it in the sense of distributions.
(3) Defined weak derivatives and gave examples/non-examples.
(4) Defined mollification of a function.

2. Weak solutions and Sobolev spaces

(1) **Weak solutions** :
Suppose $f : U \to \mathbb{R}$ is locally integrable. Define f^ϵ on U_ϵ (its "mollification") to be $f^\epsilon = \eta_\epsilon * f = \displaystyle\int_U \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy$. This operation is something like a weighted average of the values of f near x. So it "smooths out" f. The following are important properties of mollifiers.
(a) f^ϵ is smooth
(b) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.
(c) If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact sets.
(d) If $f \in L^p_{loc}(U)$ then $f^\epsilon \to f$ in $L^p_{loc}(U)$ when $1 \leq p < \infty$.
The proofs are as follows.
(a) $f^\epsilon = \displaystyle\int_U \eta_\epsilon(x-y)f(y)dy$. If we can take the derivatives inside the integral sign, then indeed f^ϵ will be smooth. We can do so by the dominated convergence theorem. Indeed, $\lim_{h_1\to 0} f^\epsilon(x+(h_1,0,\ldots))-f^\epsilon(x) = \lim_{h_1\to 0}\displaystyle\int_U \frac{\eta_\epsilon(x+(h_1,0,0..))-y)-\eta_\epsilon(x-y)}{h}f(y)dy.$
If we choose any sequence $h_{1n} \to 0$, then since $|\frac{\eta_\epsilon(x+(h_1,0,0..))-y)-\eta_\epsilon(x-y)}{h}| \leq C$ (by the mean value theorem), we see by DCT that the limit can be taken inside the integral. This argument proves that all the partial derivatives exist. By DCT we can show that these partials are also continuous. Continuing inductively, this shows that f^ϵ is smooth.
(b) The key trick behind all these convergence proofs in mollification is this : $f(x) = \displaystyle\int_{B(0,\epsilon)} \eta_\epsilon(y)f(x)dy$. So

$$|f^\epsilon(x) - f(x)| = |\int_{B_{0,\epsilon}} \eta_\epsilon(y)(f(x-y)-f(x))dy| \leq \int_{B_{0,\epsilon}} |\eta_\epsilon(y)(f(x-y)-f(x))|dy$$

Now $|\eta_\epsilon(y)| \leq \frac{C}{\epsilon^n}$. Moreover, $vol(B_\epsilon) = C\epsilon^n$. Therefore,

$$|f^\epsilon(x) - f(x)| \leq C\frac{\int_{B(0,\epsilon)} |f(x-y) - f(x)|dy}{vol(B(0,\epsilon))} = C\frac{\int_{B(x,\epsilon)} |f(z) - f(x)|dz}{vol(B(x,\epsilon))}$$

. By the Lebesgue differentiation theorem, the right hand side goes to 0 almost everywhere as $\epsilon \to 0$. This so-called Lebesgue differentiation theorem holds even in L^p (i.e. as $\epsilon \to 0$, if $f \in L^p_{loc}(U)$ $(1 \leq p < \infty)$, then $\frac{\int_{B(x,\epsilon)} |f(z) - f(x)|^p dz}{vol(B(x,\epsilon))} \to 0$ a.e. in x). This is a generalisation of the fundamental theorem of calculus.

(c) Suppose K is a compact set in U. Since $f \in C(U)$, it is uniformly continuous on K. Therefore, for every given $\epsilon > 0$, there is a $\delta > 0$ such that $K \subset U^\delta$ and if $|y| < \delta$, then $|f(x-y) - f(x)| < \epsilon \ \forall \ x \in K$. As before,

$$|f^\delta(x) - f(x)| \leq \int_{B_{0,\delta}} |\eta_\delta(y)(f(x-y) - f(x))|dy < \epsilon \ \forall \ x \in K \tag{2.1}$$

This means that $f^\delta(x)$ converges uniformly to $f(x)$.

(d) This proof is omitted and is there in an appendix of Evans.

(2) **Sobolev norm** : Note that if f is a multiply periodic function such that $f \in L^2$ and $\hat{f}(0) = 0$, then $u \in L^2$ defined by $\hat{u} = -\frac{\hat{f}}{k^2}$ is much better than L^2. In fact, $\sum_{k=-\infty}^{\infty} (1 + |k|^2)^2 |\hat{u}|^2 \leq C(\sum |\hat{f}|^2 + \sum |\hat{u}|^2) = C(\|f\|^2_{L^2} + \|u\|^2_{L^2})$. If we are in 1-dimension, then actually, $\sum_{k\neq 0} |\hat{u}| \leq \sum |k\hat{u}| = \sum |\frac{\hat{f}}{|k|}| \leq \sqrt{\sum \frac{1}{k^2}}\sqrt{\sum |\hat{f}|^2} < \infty$. So by the Weierstrass M-test, $u \in C^1$ and the Fourier series of u, u' converge uniformly to them.

In other words, the estimate $\sum_{k=-\infty}^{\infty} (1 + |k|^2)^2 |\hat{u}|^2 \leq C(\|f\|^2_{L^2} + \|u\|^2_{L^2})$ seems to imply that u is much nicer than simply being L^2.

Definition 2.1. So we define a norm called the H^s ($s > 0$ is a real number) Sobolev norm for functions $u \in L^2(S^1 \times S^1 \ldots)$ as $\|u\|_{H^s} = \|(1 + |k|^2)^{s/2}\hat{u}\|_{l^2}$.

We have the following useful lemma.

Lemma 2.2. *On the subspace of smooth and periodic functions, the following norm is equivalent to the Sobolev norm (whenever s is a non-negative integer) :*

$$\|u\|^2_{W^{s,2}} = \int_0^{2\pi} \int_0^{2\pi} \ldots (|u|^2 + |Du|^2 + |D^2u|^2 + \ldots + |D^s u|^2)$$

where $|D^s u|^2 = \sum_I |\frac{\partial^{i_1 i_2 \ldots i_s} u}{\partial x_{i_1} \partial x_{i_2} \ldots}|^2$

Proof. Note that for smooth functions, $\widehat{(D^\alpha u)} = i^{\alpha_1 + \alpha_2 + \ldots} k_1^{\alpha_1} k_2^{\alpha_2} \ldots \hat{u}$. Using this and the Parseval-Plancherel theorem the result is easily seen. $\square$

Remark 2.3. We might find it useful for later purposes to define the $W^{k,p}$ (where $1 \leq p < \infty$) norm on smooth functions defined on an arbitrary open set $U \subset \mathbb{R}^n$ (even if they are not periodic) : $\|u\|^p_{W^{k,p}} = \int_U \ldots (|u|^p + |Du|^p + |D^2u|^p + \ldots + |D^s u|^p)$. The space $W^{k,p}(U)$ is defined as the space of all locally integrable functions with p weak derivatives such that $\|u\|_{W^{k,p}} < \infty$. It turns out that it is a Banach space and that smooth functions are dense in it.

Not every L^2 function has finite Sobolev norm.

Definition 2.4. We define the Sobolev space H^s as the subspace of $L^2(S^1 \times S^1 \ldots S^1)$ of functions having finite Sobolev norm. Equip this subspace with the Sobolev inner product : $\langle u, v\rangle_{H^s} = \sum_{\vec{k}\in\mathbb{Z}^n} (1 + |k|^2)^s \hat{u}(\vec{k})\hat{v}(\vec{k})$.

It is clear that $C^s \subset H^s$.

Theorem 2.5. *Assume $s \geq 0$.*

(a) *H^s is a Hilbert space.*

(b) *Smooth functions are dense in H^s in the Sobolev norm.*

Proof. (a) If f_n is a Cauchy sequence in H^s, then $(1+|k|^2)^{s/2}\hat{f}_n$ is Cauchy in l^2. Therefore, by completeness of l^2, it converges to a_k in l^2. The sequence $b_k = \frac{a_k}{(1+|k|^2)^{s/2}}$ defines an L^2 function $f = \sum b_{\vec{k}} e^{i\vec{k}.\vec{x}}$. Clearly f is in H^s and $f_n \to f$ in H^s.

(b) Let $f_n = \sum_{|k|\leq n} \hat{f}(k)e^{i\vec{k}.\vec{x}}$. Clearly f_n is smooth. Now, $\|f_n - f\|^2_{H^s} = \sum_{|k|>n} (1+|k|^2)^s|\hat{f}(k)|^2$. Since the Sobolev norm of f is finite, as $n \to \infty$, the right hand side goes to 0.

□

The Sobolev space satisfies some other nice properties (they are collectively called the Sobolev embedding theorem).

(a) $H^s \subset H^l$ if $l \leq s$.

(b) If $s \geq [\frac{n}{2}] + 1 + a$, then $H^s \subset C^a$.

Proof. (a) Obvious.

(b) Note that $\sum_k (|k|^2+1)^{a/2}|\hat{u}(k)| \leq \|u\|_{H^s}\|(1+|k|^2)^{a/2-s/2}\|_{l^2} < \infty$ by the Cauchy integral test. Therefore, $\sum_k |k|^l|\hat{u}(k)| < \infty \ \forall\ l \leq a$. This means by the M-test that $\sum (ik)^\alpha \hat{u}(k)$ converges uniformly to some continuous functions u_α for all multiindices α with $\alpha_1 + \alpha_2 + \ldots \leq l$. By the fundamental theorem of calculus it is easily seen that $u_\alpha = D^\alpha u_0$. Thus $u_0 \in C^a$.

□

Now we define a useful notion from functional analysis.

Definition 2.6. Suppose B_1, B_2 are two Banach spaces. Then a bounded linear map $K : B_1 \to B_2$ is called compact if for every bounded sequence $x_k \in B_1$, $K(x_k)$ has a convergent subsequence in B_2.

If H_1, H_2 are two Hilbert spaces, then a bounded linear map $T : H_1 \to H_2$ is called weakly compact if for every bounded sequence $x_k \in H_1$, there is a subsequence x_{n_k} and a $u \in H_2$ so that for every $v \in H_2$, $\langle T(x_{n_k}), v\rangle \to \langle u, v\rangle$.

Sobolev spaces give nice examples of compact operators. The point is that it in the study of PDE, we want to produce "weak" solutions and then prove that secretly, these weak solutions are actually smooth. How does one produce weak solutions ? One needs something out of nothing. Such "something out of nothing" theorems are provided by functional analysis for not necessarily the PDE we are trying to solve, but for a slightly different PDE. Then one tries to use the spectral theorem for compact symmetric operators to find a weak solution for the PDE we are trying to solve. This is where compactness kicks in. All of this sounds too abstract, but we will see this in practice soon enough. For now, we will state and prove this theorem (which I promise will be useful later on). This theorem basically says "If we prove estimates for a sequence of functions in one function space, then a subsequence converges in some other space".

Theorem 2.7. *The following inclusions are compact. (Sometimes, this along with the above theorem are referred to as the Sobolev embedding theorems.)*

(a) $H^s \subset H^l$ *if* $l < s$. *(Rellich lemma.)*

(b) $H^s \subset C^a(S^1 \times S^1 \ldots)$ *if* $s \geq [\frac{n}{2}] + a + 1$ *where* C^a *is the space of* C^a *functions with the norm* $\|f\| = \max_{S^1 \times S^1 \ldots} |f(x)| + \max |Df| + \ldots + \max |D^a f|$. *(Rellich-Kondrachov compactness.)*

(c) *Suppose* U *is a bounded domain in* $\mathbb{R}^n$, *then* $C^{k,\alpha}(\bar{U}) \subset C^{k,\beta}(\bar{U})$ *if* $\beta < \alpha$ *and* $C^k \subset C^l$ *if* $l < k$. *(The Hölder space* $C^{k,\alpha}(\bar{U})$ *consists of* $C^{k,\alpha}$ *functions with the norm* $\|f\| = \max_{\bar{U}} |f| + \max |Df| + \ldots + \max |D^k f| + \sum_{|\alpha|=k} \sup_{x,y \in \bar{U}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\alpha}$. *This space is a Banach space.)*