HW 3 (due on 15th Feb (Thursday) in the class)

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

Suppose (V, ∇^v), (W, ∇^w) are two bundles with connections on a smooth manifold M. Let $\Gamma(V)$, $\Gamma(W)$, $\Gamma(V \otimes W)$ be the spaces of smooth sections. Let $S \subset \Gamma(V \otimes W)$ be the subspace consisting of sections of the form $\sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j$ where $s_i \in \Gamma(V), t_j \in \Gamma(W)$ and N is arbitrary, i.e., S consists of finite linear combinations of decomposable sections. Let X be a smooth vector field.

1. Define $\nabla_X^{v\otimes w} \sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j = \sum_{i=1,j}^{N} c_{ij} (\nabla_X^v s_i \otimes t_j + s_i \otimes \nabla_X^w t_j)$. Prove that it is linear on S, tensorial in X, and satisfies the Leibniz rule.
2. Prove that if $\sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j = \sum_{\alpha=1,\beta=1}^{M} \tilde{c}_{\alpha\beta} \tilde{s}_\alpha \otimes \tilde{t}_\beta$ on a neighbourhood U_p of a point p, then $\nabla_X^{v\otimes w}(\sum_{i=1,j}^{N} c_{ij} s_i \otimes t_j) = \nabla_X^{v\otimes w} \sum_{\alpha=1,\beta=1}^{M} \tilde{c}_{\alpha\beta} \tilde{s}_\alpha \otimes \tilde{t}_\beta$. (Hence $\nabla^{v\otimes w}$ is well-defined on S.)
3. Given a point $p \in M$, prove that if $s \in \Gamma(V \otimes W)$, then there exists a section $\tilde{s} \in S$ such that $s = \tilde{s}$ on a neighbourhood U_p of p. Conclude that $\nabla^{v\otimes w}$ is a connection on $V \otimes W$.

Comment : The same proof (almost word-to-word) shows that $d^\nabla : \Gamma(\Omega^r(M) \otimes V) \to \Gamma(\Omega^{r+1}(M) \otimes V)$ is well-defined.

Optional question : Suppose M is compact. Show that $S = \Gamma(V \otimes W)$.