HW 4 (due on 24th March (Thursday) in the class)

Please write your answers clearly and rigorously. Write your name in plain lettering (as opposed to cursive) and also staple all the pages.

1. Suppose $M(n,\mathbb{R})$ is the space of $n \times n$ matrices with coefficients in a commutative algebra R over the complex numbers $\mathbb{C}$. Assume that $\phi : M(n,R) \times M(n,R) \ldots \times M(n,R) \to \mathbb{C}$ is an invariant symmetric multilinear polynomial, i.e., $\phi(gA_1g^{-1}, gA_2g^{-1}, \ldots) = \phi(A_1, A_2, \ldots) \,\forall\, A_i \in M(n,\mathbb{R}), g \in GL(n,\mathbb{C}), \phi(A_{\sigma(1)}, A_{\sigma(2)}, \ldots) = \phi(A_1, A_2, \ldots)$ for any permutation σ, and ϕ is multilinear over $\mathbb{C}$.

   (a) Prove that there exists some integer l and $\phi : M(n,R)^l \to \mathbb{C}$ such that $\phi(A, A, A, \ldots) = \det(A)$ (where $\det(A)$ is defined as the usual determinant polynomial). Likewise, prove that there exists such a ϕ_k (and a corresponding l_k of course) for A^k for any $k \geq 1$.

   (b) Take R to be the algebra of complex-valued smooth differential forms over a smooth manifold M (where multiplication in the algebra is wedge product). Now suppose (V, ∇) is a vector bundle with a connection on M. Let F be the curvature endomorphism. Locally, F_α is a matrix of 2-forms defined as $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$. Prove that $\phi(\frac{\sqrt{-1}F_\alpha}{2\pi}, \frac{\sqrt{-1}F_\alpha}{2\pi}, \frac{\sqrt{-1}F_\alpha}{2\pi}, \ldots)$ is a globally defined $2k$-form ω_∇, i.e., it does not depend on the trivialisation chosen.

   (c) Using normal trivialisations, show that the above $2k$-form ω_∇ is closed.

   (d) Suppose ∇_1, ∇_2 are two connections. Show that $\nabla_t = t\nabla_1 + (1-t)\nabla_2$ is a connection for all $t \in [0,1]$.

   (e) Prove that the form ω_{∇_t} depends polynomially in t.

   (f) Prove that $d\frac{\omega_{\nabla_t}}{dt} = k\phi(\frac{\sqrt{-1}}{2\pi}\frac{d(dA_\alpha)}{dt}) + \frac{dA_\alpha}{dt} \wedge A_\alpha + A_\alpha \wedge \frac{dA_\alpha}{dt}, \frac{\sqrt{-1}F_\alpha}{2\pi}, \frac{\sqrt{-1}F_\alpha}{2\pi}, \ldots)$.

   (g) Prove that the form $\eta_{\nabla_t} = k\phi(\frac{dA_\alpha}{dt}, \frac{\sqrt{-1}F_\alpha}{2\pi}, \ldots)$ is a globally well-defined (i.e., independent of the trivialisation chosen) $2k-1$-form.

   (h) Using normal trivialisations for ∇_t, prove that $d\eta_{\nabla_t} = d\frac{\omega_{\nabla_t}}{dt}$.

   (i) Conclude that the De Rham cohomology class of ω_∇ is independent of ∇ (and is hence an invariant of the vector bundle).

   Remark : For ϕ corresponding to $\det(I + A)$, these classes are called the Chern classes. For ϕ corresponding to $\exp(A)$, they are called the Chern characters. For ϕ corresponding to $\det(\frac{A}{1-e^{-A}})$, they are called the Todd classes. There many other approaches to the theory of these "characteristic classes". This approach was the original one and is called Chern-Weil theory.

2. Suppose (M, g) is a compact oriented Riemannian manifold. Let η be a 1-form.

   (a) Prove that

   $$\Delta_d \eta = \nabla^* \nabla \eta + Ric_{ki} g^{ij} \eta_j dx^k \tag{1}$$

   (b) If $Ric_{ki}(p)$ is a (strictly) positive-definite matrix for all $p \in M$, then conclude that every Harmonic 1-form η is 0.

   (c) Using the Hodge theorem, conclude that $H^1(M, \mathbb{R}) = 0$ in this case.

   Remark : This conclusion (due to the Bochner technique) is a very special case of the Bonnet-Myers theorem. According to that theorem, the fundamental group is finite. Hence, the first singular homology is trivial. By the universal coefficients theorem, the first singular cohomology is trivial. It turns out that the singular and De Rham cohomology groups coincide (the De Rham theorem). Hence, the first De Rham group is trivial.