## NOTES FOR 10 MARCH (TUESDAY)

### 1. Recap

(1) Proved the regularity theorem for elliptic operators (based on some HW exercises).
(2) Proved that elliptic operators are Fredholm.

### 2. Elliptic operators - Diagonalisability

Suppose $L$ is elliptic (between the space of sections of a real bundle $\Gamma(E)$ and itself (the complex case is not very different)) and symmetric of order $2o$ satisfying Garding's coercivity inequality : $(Lu,u)_{L^2}+\lambda(u,u)_{L^2} \geq \delta(u,u)^2_{H^o}$ (for some positive $\lambda$) for all smooth sections. Also assume that $C\|u\|^2_{H^o} \geq B[u,u] = (Lu,u)_{L^2}+\lambda(u,u)_{L^2} \geq \delta\|u\|^2_{H^o}$. Let $u \in H^o$ and $u_n \to u$ in $H^o$. Since $B[u_n-u_m, u_n-u_m] \leq C\|u_n-u_m\|^2_{H^o}$, we see by completeness of reals that $B[u,u] = \lim_{n\to\infty} B[u_n,u_n]$ exists. We can also prove that $B[v_n,v_n] \to B[u,u]$ if $v_n$ is any other sequence converging to $u$. Indeed, the following little inequality is crucial for this claim and everything else that follows. Let $u,v$ be smooth sections.

$$B[u,v] = \|u\|_{H^o}\|v\|_{H^o}B[\frac{u}{\|u\|_{H^o}}, \frac{v}{\|u\|_{H^o}}]. \tag{2.1}$$

Now assuming without loss of generality that $\|u\|_{H^o} = \|v\|_{H^o} = 1$, we see by the polarisation identity and the inequalities satisfied by $B$ that $B[u,v] \leq C$. Hence, $B[u,v] \leq C\|u\|\|v\|$.
Using approximation, we can see that the extension of $B$ to $H^o$ is bilinear. Also, $B[u+v,u+v]-B[u,u]-B[v,v] = \lim_{n\to\infty} B[u_n,v_n]+B[v_n,u_n] = 2B[u,v]$ and hence $B$ remains symmetric when extended to $H^o$. Clearly, $B$ still satisfies the above inequalities.

Since $B$ is symmetric, $B[u,v]$ is a new inner product on $H^o$ which is equivalent to the Sobolev norm and hence Riesz representation implies that for every $f \in L^2$, there is a $u \in H^o$ such that $B[u,v] = (f,v)_{L^2}\ \forall\ v \in H^o$.

Suppose $f \in L^2$ and $B[u,v] = (f,v)_{L^2}\ \forall\ v \in H^o$. Thus, if $v$ is smooth, then $(u, Lv+\lambda v) = (f,v)$. Thus, $u \in H^o$ is a distributional solution to $Lu+\lambda u = f$. Hence it is smooth if $f$ is so.

Define the operator $f \in L^2 \to u_f \in H^o \subset L^2$. This is a compact operator.

**Lemma 2.1.** *The operator $K(f) = u_f$ is self-adjoint.*

*Proof.*

$$(v,Kf) = B[Kv,Kf] = B[Kf,Kv] = (f,Kv) \tag{2.2}$$

□

Hence by the spectral theorem, its spectrum consists only of countably many eigenvalues, each eigenspace is finite dimensional, and its eigenvalues are bounded above with 0 as the only accumulation point and its eigenvectors span all of $L^2$. Moreover, $K-\mu I$ is an isomorphism unless $\mu$ is an eigenvalue. Also, by Fredholm theory, $(K-\mu I)u = f$ has a solution if and only if $f$ is orthogonal to the kernel of $K^*-\mu I = K-\mu I$. Using this, here is an exercise :
Exercise : Prove that for $L$ as above, the spectrum of $L$ consists only of eigenvalues (going off to $\infty$) such that the eigenspaces are finite dimensional and span all of $L^2$.

In the case of the Hodge Laplacian, in normal coordinates one can easily see that $\Delta_d = \nabla^* \nabla + \textit{lower order terms}$. Thus, $(\Delta_d u, u) = (\nabla u, \nabla u) + (loweru, u)$. Now $|(loweru, u)| \leq C\|\nabla u\|_{L^2}\|u\|_{L^2} \leq \frac{1}{2}\|\nabla u\|^2_{L^2} + C_1\|u\|^2_{L^2}$. Hence, $(\Delta_d u, u) + (C_1 + \frac{1}{2})(u, u) \geq \frac{1}{2}(\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2})$. This proves the Garding coercivity inequality and hence the Hodge theorem. (Actually, all we need is the diagonalisability part because the rest of Hodge follows from the Fredholmness of the elliptic operator $\Delta_d$.)[1] This result opens up a wide area of study (spectral geometry). You can read from Kac's paper "Can you hear the shape of a drum ?"

## 3. Schauder and $W^{k,p}$ estimates

As in the case of $H^s$ and $C^{k,\alpha}$ we can define the $W^{k,p}(M, E)$ spaces either globally using connections (using either weak derivatives or as the completion of smooth sections) or by partitions-of-unity and the local definition.

If $L$ is elliptic (with smooth coefficients) and $u \in L^p$ is a distributional solution of $Lu = f$ where $f \in W^{k,p}$ (and $\theta$ is the order of $L$), then $u \in W^{k+\theta,p}$ with $\|u\|_{W^{k+\theta,p}} \leq C_{k,p}(\|f\|_{W^{k,p}} + \|u\|_{L^p})$. Likewise, if $f \in C^{k,\alpha}$ and $u \in C^\theta$ is a solution, then $u \in C^{k+\theta,\alpha}$ with $\|u\|_{C^{k+\theta,\alpha}} \leq C_{k,\alpha}(\|f\|_{C^{k,\alpha}} + \|u\|_{C^0})$ (the Schauder estimates). (These things are in L. Nicolaescu's lectures on the geometry of manifolds.) We shall not prove these results. The Schauder estimates are not too hard to prove but the $W^{k,p}$ estimates require some heavy harmonic analysis (the Calderon-Zygmund inequality).

[1] Actually, since lower order terms do not make a difference to Fredholmness, this also proves that the above kind of operators plus lower order terms are still Fredholm.