NOTES FOR 12 MARCH (THURSDAY)

1. Recap

(1) Proved that strongly elliptic operators are diagonalisable.
(2) Stated other regularity results.

2. Parabolic equations

Let L be an order 2θ elliptic formally self-adjoint operator satisfying the Garding coercivity inequality $(Lv, v)_{L^2} \geq \delta(v, v)_{H^\theta}$ such that $L : \Gamma(E) \to \Gamma(E)$ and let u_0, f be smooth sections of E. Then the equation $\frac{du}{dt} = -Lu + f,\ u(0) = u_0$ for a section $u : [0, T] \times M \to E$ is called a linear parabolic PDE. The quintessential example of a parabolic PDE is the heat equation $\frac{du}{dt} = \Delta u$. (The equation $\frac{du}{dt} = -\Delta u$ is called the backwards heat equation and is usually badly behaved.)

We typically want u to be smooth on the interior of the parabolic domain and smooth from the right hand side at $t = 0$.

Theorem 2.1. *Every parabolic equation has a unique smooth solution for all time, i.e., on* $[0, \infty)\times M$.

Proof. First we prove uniqueness. Indeed, if there are two solutions, then let $v = u_1 - u_2$. It satisfies $\frac{dv}{dt} = -Lv, v(0) = 0$. Now,

$$\frac{d(v, v)_{L^2}}{dt} = -2(Lv, v) \leq -\delta(v, v)_{L^2}. \tag{2.1}$$

Hence,

$$(v, v)(t) \leq (v, v)(0)e^{-\delta t}.$$

Thus $v \equiv 0$. The estimate on v (an "Energy estimate") is useful in its own right. One can similarly prove that if $\frac{dv}{dt} = -Lv + f$, then $(v, v)(t) \leq C(1 + t)$.

Now we prove existence. Let e_n be a countable family of smooth eigenvectors with eigenvalues $\lambda_n > 0$ of L spanning L^2. Thus, $u_0 = \sum_n c_n e_n$ for any $u_0 \in L^2$ (and $f = \sum_n f_n e_n$). Since $u_0 \in L^2$, we see that $\sum_n |c_n|^2 < \infty$. First we prove that the quantity $\|u_0\|_k = \sum_n |c_n|^2(1 + \lambda_n)^{2k}$ is equivalent to the $H^{k2\theta}$ norm. Indeed, if $\|u_0\|_k < \infty$, then $(u_0, L^k e_n)_{L^2} = \lambda_n^k c_n$. If ϕ is a smooth section, then $\phi = \sum_n \phi_n e_n$. Thus, $L^k\phi \in L^2$ satisfies $(L^k\phi, e_n) = \phi_n\lambda_n^k$. Therefore, $(u_0, L^k\phi) = \sum_n c_n\lambda_n^k\phi_n$ and hence $L^k u_0 = f_k$ in the sense of distributions where $f_k \in L^2$. Therefore, $u_0 \in H^{k\theta}$ and $\|u_0\|_{H^{k2\theta}} \leq C_k\|u_0\|_k$. Conversely, if $u_0 \in H^{2k\theta}$, then $\|L^k u_0\|_{L^2} \leq C\|u_0\|_{H^{2k\theta}} < \infty$. Thus, $(L^k u_0, e_n) = (u_0, L^k e_n) = \lambda_n^k c_n$. Therefore, $\|u_0\|_k < \infty$ and $\|u_0\|_k^2 \leq \tilde{C}_k\|u_0\|^2_{H^{2k\theta}}$.

Define the function $u(t) = \sum_n c_n e^{-\lambda_n t} e_n + \frac{f_n}{\lambda_n}(1 - e^{-\lambda_n t})e_n$. Clearly $u(t) \in L^2$. Moreover, $\|u(t) - u_0\|^2_{L^2} = \sum_n |c_n|^2(1 - e^{-\lambda_n t})^2$ which by DCT converges to 0 as $t \to 0^+$.

Now we proceed to prove that $u(t, x)$ is C^∞ in x for every fixed $t \geq 0$ and that we can differentiate w.r.t x term-by-term. Since $\sum_n c_n e_n$ and $\sum_n f_n e_n$ are smooth (by assumption), their $\|.\|_k$ norms are finite for all (by the equivalence of norms above). Therefore, $\|u\|_{H^{2k\theta}} \leq C_k\ \forall\ k$. Hence u is smooth in x for all fixed $t \geq 0$. Moreover, by Sobolev embedding, the partial sum $s_N(t) = \|\sum_{n=1}^N u_n e_n\|_{C^{k,\alpha}} \leq \tilde{C}_k$ independent of N. Therefore, by Arzela-Ascoli, every subsequence has a subsequence that converges in C^l and in fact the limits are all $u(t)$ because $s_N(t) \to u(t)$ in L^2.

Therefore, $u(t) \in C^l$ for all l and the term-by-term derivatives in x converge.

Now note that if $u(t) = \sum_n u_n(t)e_n$ where $\|s_N(t)\|_{H^k} \leq C_k$ independent of $N, t \geq 0$, then $\|s_N(t) - s_N(t_0)\|_k^2 \leq \sum_{n=1}^N (1+\lambda_n)^{2k}(2\lambda_n^2|c_n|^2 + \lambda_n^2|f_n|^2)|t-t_0|^2 \leq C_k$ and hence $\|s_N(t) - s_N(t_0)\|_{C^0} < \epsilon$ for $t - t_0$ small (if $t_0 = 0$, then $t \geq 0$). So $u(t,x)$is continuous in (t,x). Actually, this argument shows that $\partial_x^l u(t,x)$ is continuous too.

Likewise, $\|s'_N(t) - s'_N(t_0)\|_{C^0} < \epsilon$ for t close to t_0. So the term-by-term derivatives $s'_N(t)$ converge uniformly to a continuous function $v(t,x)$. Note that $\int_0^t v(a)da = \lim_{N\to\infty} \int_0^t s'_N(a)da = u(s)$ and hence by the FTC, $u'(t,x) = v(t,x)$ and moreover, $u'(t,x)$ is continuous in t, x. (Actually it shows that all the partials in x are also continuous.) Inductively, we can prove that u is smooth on $[0,\infty)\times M$ and that we can differentiate term-by-term.

Finally, an easy calculation shows that u satisfies the equation with the boundary conditions. $\square$

3. Uniformisation theorem

A natural question in Riemannian geometry is the Yamabe problem : Given a compact oriented (M, g_0), find a smooth function $f : M \to \mathbb{R}$ so that $(M, g = e^{-f}g_0)$ has constant scalar curvature. When M is 2-dim, the scalar curvature is upto a factor, the Gaussian curvature K. In such a case, $\int K dA = 2\pi\chi(M)$ (the Gauss-Bonnet theorem) and hence the constant is fixed by the topology of the manifold. So in 2-dim, the equation is strongly linked to the topology of the manifold. This problem is called the Riemannian uniformisation problem.