NOTES FOR 13 FEB (THURSDAY)

1. Recap

(1) Defined the Hodge star and proved some properties.
(2) Defined the codifferential and the formal adjoint of the connection operator.
(3) Defined the Hodge Laplacian and calculated it for the Euclidean space. Defined the Rough Laplacian for sections of vector bundles.

2. Divergence, Stokes' theorem, and Laplacians

Suppose we take $E = \Omega^k(M)$, then potentially, we have two Laplacians, Δ_d and $\nabla^*\nabla$. It turns out that

$$\Delta_d \eta = \nabla^*\nabla\eta + Curvature(\eta) \tag{2.1}$$

where the last term is something that depends linearly on η with coefficients depending on the Riemann tensor. This sort of an identity relating two different Laplacians is called a Bochner-Weitzenböck identity. So, taking inner product with η and integrating,

$$(d\eta, d\eta) + (d^\dagger\eta, d^\dagger\eta) = (\nabla\eta, \nabla\eta) + (\eta, Curvature(\eta)) \geq (\eta, Curvature(\eta)) \tag{2.2}$$

So if $\Delta_d \eta = 0$, i.e., η is Harmonic, and the curvature term is positive, we have a contradiction unless $\eta = 0$. This sort of a conclusion turns out to be useful for topology. This method is called the Bochner technique for proving non-existence of non-trivial Harmonic objects.

3. Statement of the Hodge theorem and applications

To calculate the De Rham cohomology groups $H^k(M)$, it is useful to have "good, canonical" representatives of each cohomology class. That is, given a class $[\eta]$ consisting of forms $\alpha = \eta + d\gamma$, we want to find the "best" possible $\alpha \in [\eta]$. More precisely, we want to minimise the "energy" $E_g(\alpha) = \int_M |\alpha|_g^2 vol_g$. Suppose α is such a smooth minimiser. Then $\frac{dE(\alpha+td\gamma)}{dt}|_{t=0} = 0 \ \forall \ \gamma$.

$$\frac{dE(\alpha + td\gamma)}{dt}|_{t=0} = \int_M 2\langle d\gamma, \alpha\rangle = 2(d\gamma, \alpha) = 0 \Leftrightarrow (\gamma, d^\dagger\alpha) = 0 \Leftrightarrow d^\dagger\alpha = 0 \tag{3.1}$$

This means that $d\alpha = 0 = d^\dagger\alpha \Rightarrow \Delta_d\alpha = 0$. Conversely, if $\Delta_d\alpha = 0$, then taking inner product with α we see that $\|d\alpha\|^2 + \|d^\dagger\alpha\|^2 = 0$. This means that $d\alpha = 0 = d^\dagger\alpha$. So, ideally, we'd like a statement to the effect of:

Theorem 3.1 (Hodge's theorem). *Suppose (M, g) is compact and oriented. The space of Harmonic forms $\mathcal{H}^k$ among the space of smooth forms is finite dimensional. Therefore there is an orthogonal projection $H : Smooth\ k\ forms \rightarrow \mathcal{H}^k$ and a unique operator $G : Smooth\ k\ forms \rightarrow Smooth\ k\ forms$ such that $Gd = dG, d^\dagger G = Gd^\dagger$ and $I = H + \Delta_d G$. As a consequence, every De Rham cohomology class has a unique harmonic representative. Also, the Hodge Laplacian is diagonalisable, i.e., there is a complete orthonormal basis of eigenvectors.*

If we manage to prove this, we have some wonderful conclusions (for compact oriented manifolds):

(1) A weak form of Poincaré duality : The map $H^k(M) \times H^{m-k}(M) \to \mathbb{R}$ given by $[\omega], [\eta] \to \int_M [\omega \wedge \eta]$ is non-degenerate. Thus $dim(H^k(M)) = dim(H^{m-k}(M))$. Indeed, choose any metric on M and suppose $\omega \in [\omega]$ is harmonic, i.e., $d\omega = d^\dagger\omega = 0$. Then $*\omega$ is also harmonic because $d*\omega = \pm **d*\omega = 0$ and $d^\dagger * \omega = \pm * d *^2 \omega = \pm * d\omega = 0$. Now $\int \omega \wedge *\omega = \|\omega\|^2 = 0$ if and only if $\omega = 0$, i.e., $[\omega] = [0]$. The Poincaré duality theorem implies that $\chi(M) = dim(H^0(M)) - dim(H^1(M)) + \ldots$ is zero for odd dimensional manifolds. This $\chi(M)$ turns out to be the Euler characteristic, i.e., the alternating sum of the vertices, edges, etc if you triangulate the manifold.

(2) A weak form of the Kunneth formula : $H^k(M \times N) \simeq \oplus_{l=0}^k H^l(M) \otimes H^{k-l}(N)$ with the map being $\oplus[\omega_i] \otimes [\eta_j] \to \sum[\pi_1^*\omega_i \wedge \pi_2^*\eta_j]$. Choose metrics $g_M, g_N, g_M \times g_N$ on $M, N, M \times N$ respectively, and suppose we represent all classes with their harmonic representatives. Then $\pi_1^*\omega_i \wedge \pi_2^*\eta_j$ are harmonic with respect to the product metric. Indeed, obviously they are closed. Now

$$\begin{aligned}(d^{\dagger_{M\times N}}\pi_1^*\omega_i \wedge \pi_2^*\eta_j, \pi_1^*\alpha \wedge \pi_2^*\beta) &= (\pi_1^*\omega_i \wedge \pi_2^*\eta_j, \pi_1^*d\alpha \wedge \pi_2^*\beta \pm \pi_1^*\alpha \wedge \pi_2^*d\beta) \\ &= (\pi_1^*\omega_i, \pi_1^*d\alpha)(\pi_2^*\eta_i, \pi_2^*\beta) + (\pi_1^*\omega_i, \pi_1^*\alpha)(\pi_2^*\eta_i, \pi_2^*d\beta) = 0\end{aligned} \tag{3.2}$$

In fact, one can prove that $\Delta_{M\times N} = \Delta_M + \Delta_N$. Thus, the map at the level of harmonic forms is well-defined. It is clear that it is injective. To prove it is surjective requires some more effort. One has to identify the eigenvectors of the Laplacian and prove it consists of decomposable forms.

Indeed, first one proves that if ω_i, η_j are the orthonormal bases of eigenvectors of Δ_M, Δ_N respectively, then $\pi_1^*\omega_i \wedge \pi_2^*\eta_j$ form a complete orthonormal basis for $L^2(M \times N, g_M \times g_N)$. This can be accomplished by proving that if α is any L^2 form, then $(\alpha, \pi_1^*\omega_i \wedge \pi_2^*\eta_j) = 0$ for all i, j, then α ought to be 0. Indeed, this implies that $(\alpha, \pi_1^*\omega \wedge \pi_2^*\eta) = 0$ for all ω, η because ω_i, η_j form bases for $L^2(M, g_M), L^2(N, g_N)$. Now near a point p, α can be written as a finite sum of such decomposable forms. Using a cut-off function, one can see that $\alpha(p) = 0 \; \forall \; p$.

Second, note that $\Delta_{M\times N}(\pi_1^*\omega_i \wedge \pi_2^*\eta_j) = (\lambda_i + \mu_j)\pi_1^*\omega_i \wedge \pi_2^*\eta_j$. So $\pi_1^*\omega_i \wedge \pi_2^*\eta_j$ form a complete basis of eigenvectors for $\Delta_{M\times N}$. (A small argument shows that if you have a complete basis of eigenvectors, then every eigenvector better be one of these (or in the case of repeated eigenvalues, a linear combination of these).) Since $\lambda_i + \mu_j \geq 0$, equality holds if and only if $\lambda_i = \eta_j = 0$. Therefore, the harmonic forms of $\Delta_{M\times N}$ are obtained only this way.

For the flat torus, since we already proved that Δ_d is a constant coefficient symmetric elliptic operator, and that elliptic operators are Fredholm, we see that $\Delta_d\eta = \omega$ can be solved for η if and only if ω is orthogonal to the space of harmonic forms (which we proved is finite dimensional). Moreover, we can choose η to be the unique one having the smallest L^2-norm. So we have $\eta = G(\omega)$. Thus, every form ω can be uniquely written as $H(\omega) + \Delta G(\omega)$. Now $\Delta_d d(G\omega) = dd^\dagger dG\omega = d\Delta_d G\omega = d\omega$. This does not yet show that $d(G\omega) = G(d\omega)$. We need to show that $d(G\omega)$ has the smallest L^2-norm among all such solutions, i.e., it is orthogonal to harmonic forms. But indeed, $(d(G\omega), \alpha) = (G\omega, d^\dagger\alpha) = 0$. Likewise, G commutes with $d^\dagger$. As for the completeness of the eigenfunctions, one can explicitly calculate these eigenfunctions as simply being of the form e^{ikx}. We know that the Fourier functions are complete in L^2 (Parseval-Plancherel).

Seeing how useful this Hodge theorem is, we want to prove it for general compact oriented (M, g). There are several approaches to this. One is to prove such a result for general elliptic operators.

(However, that approach has the disadvantage that it does not say much about eigenvalues. So we have to deal with that issue.)

4. Sobolev spaces on general manifolds

The theory of Sobolev spaces, Sobolev embedding, etc goes over to general manifolds. We will focus on that now.

There are many ways of defining $H^s(M,E)$:

Definition 4.1. Suppose (E,h,∇) is a vector bundle with metric and connection on a compact oriented (M,g) and $s \geq 0$ is an integer. Suppose t is a smooth section of E. Define $\|t\|^2_{H^s} = \int_M (|t|^2 + |\nabla t|^2 + \ldots + |\nabla^s t|^2) vol_g$. Define $H^s_{\nabla,h,g}$ to be the completion of this space (in the metric space sense). Concretely, H^s consists of L^2 sections t such that there exist smooth sections $t_n \to t$ in L^2 and t_n form a Cauchy sequence in the H^s norm.

The claim is that these spaces are equivalent. Indeed,

Lemma 4.2. *The Sobolev norms are equivalent (on smooth sections) for different h, ∇, g.*

Proof. Suppose we choose $h_1, \nabla_1, g_1,\ h_2, \nabla_2, g_2$. First of all, it is easy to see that there exists a positive finite constant C so that $\frac{1}{C}h_1 \leq h_2 \leq Ch_2$, $\frac{1}{C}g_2 \leq g_1 \leq Cg_2$ where the inequalities are in the sense of positive-definite matrices. Now $\nabla_1 = \nabla_2 + B$ where B is an endomorphism of E. Let $|B|_1, |B|_2 \leq C$. Now $\frac{1}{C^2}|\nabla_1 t|_{h_2 \otimes g_2} \leq |\nabla_1 t|^2_{h_1 \otimes g_1} \leq C^2 |\nabla_1 t|^2_{h_2 \otimes g_2}$. Now $|\nabla_1 t|_{h_2 \otimes g_2} \leq |\nabla_2 t|_2 + C|t|_2$. Moreover, $|\nabla_2 t|_2 \leq |\nabla_1 t|_2 + C|t|_2$. Hence, $\frac{1}{K}(|t|^2_2 + |\nabla_2 t|^2_2) \leq |t|^2_2 + |\nabla_1 t|^2_2 \leq K(|t|^2_2 + |\nabla_2 t|^2_2)$. By induction, we can show this for all derivatives. $\square$

Remark 4.3. Note that the above proof works even for open subsets U of a compact manifold M.