NOTES FOR 14 JAN (TUESDAY)

1. Recap

(1) Proved some properties of mollifiers (Evans' appendix).
(2) Defined the Sobolev norm and the Sobolev space. Proved that the Sobolev space is a Hilbert space and that smooth functions are dense in it.
(3) Proved the Sobolev embedding theorem. Defined compact operators between Banach spaces.

2. Weak solutions and Sobolev spaces

Theorem 2.1. *The following inclusions are compact. (Sometimes, this along with the above theorem are referred to as the Sobolev embedding theorems.)*

(1) *$H^s \subset H^l$ if $l < s$. (Rellich lemma.)*
(2) *$H^s \subset C^a(S^1 \times S^1 \ldots)$ if $s \geq [\frac{n}{2}] + a + 1$ where C^a is the space of C^a functions with the norm $\|f\| = \max_{S^1 \times S^1 \ldots} |f(x)| + \max |Df| + \ldots + \max |D^a f|$. (Rellich-Kondrachov compactness.)*
(3) *Suppose U is a bounded domain in $\mathbb{R}^n$, then $C^{k,\alpha}(\bar{U}) \subset C^{k,\beta}(\bar{U})$ if $\beta < \alpha$ and $C^k \subset C^l$ if $l < k$. (The Hölder space $C^{k,\alpha}(\bar{U})$ consists of $C^{k,\alpha}$ functions with the norm $\|f\| = \max\limits_{\bar{U}} |f| + \max |Df| + \ldots + \max |D^k f| + \sum\limits_{|\alpha| = k} \sup\limits_{x,y \in \bar{U}} \dfrac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\alpha}$. This space is a Banach space.)*
(4) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary (in the sense that the boundary is a smooth submanifold of $\mathbb{R}^n$). Then,*
   (a) *If $p < n$, $W^{1,p}$ is compactly contained in L^q where $q < p^* = \frac{np}{n-p}$ (please note the strict inequality. I did not write it correctly in the class). The number p^* is called a critical exponent.*
   (b) *If $p > n$, then $W^{1,p}$ is compactly contained in $C^{0,\gamma}$ for some $\gamma > 0$ determined by p, n.*

Proof. (1) If f_n is a bounded sequence in H^s, then $|\hat{f}_n(\vec{k})|^2(1+|k|^2)^s$ is a bounded sequence of real numbers for all k. Enumerate $\vec{k}$ by positive integers a. Therefore, by completeness of reals, we may assume that there exists a subsequence of functions $a_{1i}(x) = f_{n_i}(x)$ such that $\hat{a}_{1i}(1)^2(1+|k|^2)^s$ converges to a real number. From this subsequence choose a further subsequence $a_{2i}(x)$ such that $\hat{a}_{2i}(1)(1+|k_1|^2)^s$, $\hat{a}_{2i}(2)(1+|k_2|^2)^s$ converge. Continue like this. Now choose the diagonal subsequence $b_i(x) = a_{ii}(x)$. It is easy to see that $\hat{b}_i(\vec{k})(1+|k_i|^2)^s$ is Cauchy for all $\vec{k}$.
Now, $\|b_i - b_j\|_{H^l}^2 = \sum\limits_{\vec{k}} |\hat{b}_i(\vec{k}) - \hat{b}_j(\vec{k})|^2 (1+|k|^2)^{l/2}$. When $|k| > N = \epsilon^{2/(l-s)}$, we see that $\sum\limits_{|k|>N} |\hat{b}_i(\vec{k}) - \hat{b}_j(\vec{k})|^2 (1+|k|^2)^{s/2} \dfrac{1}{(1+|k|^2)^{s/2-l/2}} \leq \dfrac{C}{N^{s/2-l/2}} < C\epsilon$. For the other smaller values of $|k|$, choose M is so large that that $b_i(k) - b_j(k)$ is small for all $|k| < N$ and $i,j > M$.

(2) As above, choose the subsequence $b_i(x)$. We will prove that it is Cauchy in the space C^a. If the Fourier series of $b_i - b_j$ (and its derivatives upto order a) converged to it (respectively to

its derivatives) uniformly, then,

$$\|b_i - b_j\|_{C^a} = \|\sum \widehat{b_i - b_j}(\vec{k})e^{i\vec{k}.\vec{x}}\|_{C^a} \leq \sum_{p=0}^{p=a} \sum_{\vec{k}} |\widehat{b_i - b_j}(\vec{k})||k|^p \tag{2.1}$$

As before, for $|\vec{k}| > N$, $\sum_{p=0}^{p=a} \sum_{|\vec{k}|>N} |\widehat{b_i - b_j}(\vec{k})||k|^p \leq C\|b_i - b_j\|_{H^s} \sum_{|k|>N} (1+|k|^2)^{a-s} < \epsilon$ for some large N. For $|\vec{k}| \leq N$, as before, we can choose M so that $i, j > M$ implies that the finitely many terms are small.

Now, by the Weierstrass M-test, indeed the Fourier series of $b_i - b_j$ converges uniformly to it (and likewise for its derivatives). So the above argument shows that $\|b_i - b_j\|_{C^a} < \epsilon$ if $i, j > N$.

(3) HW
(4) Omitted. (Evans' book.) The first step is to prove inclusion, and then to prove compactness. Proving inclusion is tricky. (The point is to use the fundamental theorem of calculus and the Hölder inequality in a clever way.)

□

3. Constant-coefficient elliptic operators on the torus

Everything we did earlier holds true for vector-valued periodic functions, i.e., $\vec{u} : S^1 \times \ldots S^1 \to \mathbb{R}^\mu$. (By the way, these things work even when $\mathbb{R}$ is replaced by $\mathbb{C}$ on the right hand side, i.e., for complex-valued functions.) We can define a Fourier series if $\vec{u} \in L^1_{loc}$, $\widehat{\vec{u}(\vec{k})} = \frac{1}{(2\pi)^n} \int\int \ldots \vec{u}(\vec{x}) e^{-i\vec{k}.\vec{x}} d^n x$. We can define Sobolev spaces $H^s(S^1 \times S^1 \ldots, \mathbb{R}^\mu)$, and prove the Sobolev embedding and compactness theorems. The Parseval-Plancherel theorem also holds. Moreover, so does the formula relating the Fourier transform of the derivative to that of the function. (By the way, $\langle \vec{u}, \vec{v} \rangle = \sum (1+|k|^2)^s \widehat{\vec{u}}.\widehat{\vec{v}}$.)

Instead of studying $\Delta \vec{u} = \vec{f}$, let us generalise much more. Suppose we want to study

$$L(\vec{u}) = \sum_{|\alpha|=l} [A]_{l,\alpha} D^\alpha \vec{u} + \sum_{|\alpha|=l-1} [A]_{l-1,\alpha} D^\alpha \vec{u} + \ldots = \vec{f},$$

where $A_{k,\alpha}$ are $\mu \times \mu$ matrices of constants, one for each l, α such that $|\alpha| = \alpha_1 + \alpha_2 + \ldots = l$.

Now $L : H^{s+l} \to H^s$ is a bounded linear map. Take Fourier (series) transform on both sides. Now

$$(\sum_{|\alpha|=l} [A]_{l,\alpha} (ik)^\alpha + \sum_{|\alpha|=l-1} [A]_{l-1,\alpha} (ik)^\alpha + \ldots) \widehat{\vec{u}}(\vec{k}) = \widehat{\vec{f}}(\vec{k}).$$

This means that for large $|\vec{k}|$, the equation above has a solution if and only if the top order term is invertible, i.e., $\sigma_{\vec{k}} = \sum_{|\alpha|=l} [A]_{l,\alpha} (ik)^\alpha$ is an invertible $\mu \times \mu$ matrix for all $|k| \neq 0$. (Note that by homogenity, if it is invertible for all large $|k|$, then it is so for all non-zero ones.)

Definition 3.1. A linear differential operator L with constant coefficients on the torus is said to be elliptic if the principal symbol $\sigma_{\vec{k}}$ is invertible for all $|k| \neq 0$.

Assume that L is elliptic. Because the A are constants, there exist constants (called the ellipticity constants) δ_1, δ_2 such that $\delta_2 \|\vec{k}\|^l \|\vec{v}\| \geq \|[\sigma_{\vec{k}}][\vec{v}]\| \geq \delta_1 \|\vec{k}\|^l \|\vec{v}\|$ for all $\mu \times 1$ column vectors $\vec{v}$.

Even for elliptic operators, the above equation for Fourier coefficients cannot always be inverted. However, for sufficiently large $|k|$, it can be inverted to produce an "approximate" solution $\vec{u}_{app}$ whose Fourier coefficients are 0 for $|k| \leq N$ and $\widehat{\vec{u}_{app}}(\vec{k}) = \hat{L}_{\vec{k}}^{-1}\widehat{\vec{f}}(\vec{k})$. We claim that

Theorem 3.2. *If $\vec{f}$ is in H^s and L is elliptic, then*

1. $\vec{u}_{app}$ *is in* H^{s+l}.
2. *The map $G : H^s \to H^{s+l}$ given by $G(f) = \vec{u}_{app}$ is a bounded linear map (with the bound depending on the ellipticity constants, s, l, and coefficients of the lower order terms).*
3. *$L \circ G - I : H^s \to H^s$ and $G \circ L - I : H^{s+l} \to H^{s+l}$ are compact operators. (In simple english, G is an "almost" inverse of L. It is called a parametrix for L.)*
4. *If $\vec{u} \in H^{s+l}$ satisfies $L(\vec{u}) = \vec{f}$, then $\|u\|_{H^{s+l}} \leq C(\|f\|_{H^s} + \|u|_{L^2})$ where C depends only on the ellipticity constants, s, l, and bounds on the other coefficients (the lower order terms).*